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SPECTROSCOPY AND STRUCTURAL PROPERTIES OF HEAVY QUARKONIUM SYSTEMS

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Abstract. The spectroscopy and structural properties of heavy quarkonium systems are investigated within the Varshni–Hellmann potential model. The radial Schrödinger equation is solved analytically using the Nikiforov–Uvarov method to obtain the energy eigenvalues and normalized wavefunctions of charmonium and bottomonium states. The calculated mass spectra are compared with available experimental data and results from previous theoretical studies. The present model yields sum of absolute deviations (SADs) of 0.002 GeV and 0.003, with corresponding mean absolute errors of 0.00029 GeV and 0.00038 GeV for bottomonium and charmonium, respectively. The corresponding wavefunctions are examined to characterize the spatial structure of the bound states. In addition, the wavefunctions at the origin are used within the Van Royen–Weisskopf formalism to calculate the leptonic decay widths and vector decay constants of the $J/\psi(1S)$ and $Y(1S)$ states. The calculated leptonic widths are 4.467 keV for $J/\psi(1S)$ and 1.190 keV for $Y(1S)$, compared with the experimental values of 5.55 keV and 1.340 keV, respectively. The results demonstrate that the Varshni–Hellmann potential provides an analytically tractable framework for describing heavy quarkonium mass spectra, wavefunctions, and selected decay properties. The framework may provide a basis for future investigations of exotic hadronic systems and quarkonium behaviour in extreme quantum chromodynamics environments.

Keywords: Heavy Mesons; Varshni–Hellmann Potential; Quark–antiquark interactions; Schrödinger Equation; Quarkonium Spectroscopy.

1. Introduction

Quark–antiquark interactions lie at the foundation of hadron physics and quantum chromodynamics (QCD), providing the fundamental mechanism responsible for binding quarks into mesons and baryons. Heavy mesons, which are bound states of a heavy quark and its antiquark such as charmonium and bottomonium [1], serve as particularly insightful probes of the non-perturbative regime of QCD. The relatively large masses of the constituent quarks result in low internal velocities, enabling the application of non-relativistic quantum mechanics to their dynamics. This allows theoretical models based on the Schrödinger equation to capture key features of heavy mesons, including their energy spectra, decay properties, spatial structures, and transition rates [2]. Accurate modeling of these systems not only deepens our understanding of QCD in the strong-

coupling regime but also provides essential benchmarks for interpreting experimental results from high-energy physics facilities. Historically, traditional potential models have formed the backbone of quarkonium spectroscopy. Among these, the Cornell potential has been particularly influential due to its simple yet effective form, combining a Coulomb-like term at short distances with a linear confining term at long ranges [3,4]. This formulation successfully reproduces ground-state masses and low-lying excitations of charmonium and bottomonium while capturing the essential feature of confinement in QCD.

However, recent high-precision experiments at LHCb, Belle II, BESIII, and other facilities have revealed a variety of new meson states, including exotic configurations such as tetraquarks, pentaquarks, and other multiquark systems. These discoveries challenge the explanatory power of conventional potentials and highlight the need for more flexible models capable of incorporating medium effects, screening phenomena, and complex inter-quark interactions [5–7]. To address these challenges, several alternative potentials have been proposed. The Varshni potential, originally developed in molecular physics, introduces a tunable interplay between attractive and repulsive forces, making it particularly suitable for modeling intermediate-range interactions [8]. In the context of quarkonium, it captures the delicate balance between binding and repulsive effects that influence radial excitations and level spacings. The Hellmann potential, on the other hand, combines a Coulomb term with a Yukawa-type screening component, effectively describing quark–antiquark interactions in environments where screening, thermal fluctuations, or density effects modify the effective potential [9]. Both potentials extend the flexibility of conventional models, incorporating phenomena such as finite-range screening, medium-induced weakening of confinement, and modifications to excited-state energy levels. Extensive theoretical studies have demonstrated the efficacy of these exponential potentials when applied to quarkonium systems. Purohit *et al.* [10] employed the Linear plus Modified Yukawa Potential with the Nikiforov–Uvarov (NU) method to study heavy–light meson spectra, reporting good agreement with experimental data, particularly for low-lying states. Rai and Rathaud [11] employed the Hellmann potential with relativistic and one-pion exchange corrections to compute dimesonic molecular mass spectra and decay widths, reporting good agreement with experimental states. Inyang *et al.* [12] applied the Nikiforov–Uvarov (NU) method to derive analytical energy spectra for several mesonic systems, extending the approach to exponential-type potentials and reinforcing the versatility of semi-analytical methods in quarkonium spectroscopy [2,13]. Complementary calculations using Dyson–Schwinger equations by Yin *et al.* [14] provided independent evaluations of ground-state meson and baryon masses, serving as reliable benchmarks for potential-based models. Hybrid approaches, such as the modified Kratzer plus screened Coulomb potential studied by Ibekwe *et al.* [15], further demonstrated that analytical methods can produce spectra consistent with experimental observations.

The investigation of thermodynamic and medium-dependent effects has also been an active area of research. Abu-Shady and Fath-Allah [16] employed generalized fractional Klein–Gordon and Schrödinger equations to explore the influence of temperature, plasma anisotropy, topological defects, and baryonic chemical potential on charmonium and bottomonium binding energies and wavefunctions. Their results indicate that increasing temperature generally reduces binding energies, while anisotropy and topological defects can enhance binding. Fractional models consistently yield lower binding energies and suppressed wavefunctions relative to classical cases, demonstrating the potential of fractional calculus in capturing complex medium effects. Additionally, screened potentials have been widely used to study quarkonium dissociation and medium modifications, with methods such as the NU method and the Asymptotic Iteration Method (AIM) providing accurate predictions for mass spectra under varying physical conditions [17,18]. Despite these advances, a combined potential that integrates the strengths of both Varshni and Hellmann forms has not yet been systematically explored. The Varshni–Hellmann potential (VHP) [19] merges the intermediate-range flexibility of the Varshni form with the short-range Coulomb binding and long-range Yukawa screening of the Hellmann potential. This combination provides a framework to model quark confinement more accurately across all distance scales while retaining analytical tractability. Although similar hybrid potentials have been applied in information-theoretic analyses [20], their systematic application to heavy meson spectroscopy remains novel. The primary objective of this study is to investigate the Varshni–Hellmann potential (VHP) as a unified framework for describing heavy quarkonium systems. The non-relativistic radial Schrödinger equation is solved analytically using the Nikiforov–Uvarov method to obtain energy eigenvalues and wavefunctions, which are used to calculate the mass spectra, wavefunctions at the origin, leptonic widths, and vector decay constants of charmonium and bottomonium states. The results are compared with experimental data and selected theoretical predictions to assess the model’s predictive capability.

The novelty lies in the systematic application of the combined VHP to both charmonium and bottomonium spectroscopy and related S-wave decay observables. Although the Varshni and Hellmann potentials have been studied separately, and their combination has been applied to selected quantum and molecular systems, its unified application to heavy quarkonium remains limited. This study therefore provides a consistent assessment of the VHP for describing the spectroscopic and structural properties of heavy quarkonium.

The sections that follow introduce the formalism of the Varshni–Hellmann potential and outline the implementation of the NU method for deriving analytical solutions. Computed mass spectra and structural observables are subsequently presented and compared with experimental data and theoretical predictions. The work concludes with an analysis of the results and a summary of the key findings.

The VHP is of the form [19]:

$$V(r) = A_0 - \frac{A_0 B_0 e^{-\alpha r}}{r} - \frac{C_0}{r} + \frac{D_0 e^{-\alpha r}}{r}, \quad (1)$$

where A_0, B_0, C_0 , and D_0 are potential strength parameters, r is inter nuclear distance, and α is the screening parameter.

To model the potential describing the quark–antiquark interaction, we perform a series expansion of the exponential terms in Eq. (1) up to third order, which yields:

$$V(r) = -\frac{\beta_a}{r} + \beta_b r - \beta_c r^2 + \beta_d \quad (2)$$

where

$$\left. \begin{aligned} \beta_a &= A_0 B_0 + C_0 - D_0, \quad \beta_b = -\frac{A_0 B_0 \alpha^2}{2} + \frac{\alpha^2 D_0}{2} \\ \beta_c &= -\frac{A_0 B_0 \alpha^3}{6} + \frac{D_0 \alpha^3}{6}, \quad \beta_d = A_0 + \alpha A_0 B_0 - \alpha D_0 \end{aligned} \right\} \quad (3)$$

2. Analytical Treatment of the Varshni–Hellmann Potential

The Schrödinger equation (SE), which describes the quantum dynamics of a particle subject to a potential, provides the framework for calculating the energy eigenvalues and corresponding wave functions of the system. In the present study, the SE is expressed as [2]

$$\frac{d^2 \psi_{nl}(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2} (E_{nl} - V(r)) - \frac{l(l+1)}{r^2} \right] \psi_{nl}(r) = 0 \quad (4)$$

Here, ℓ and μ represent the angular momentum quantum number and the reduced mass of the quarkonium system, respectively, while r denotes the inter-particle distance and $\hbar$ is the reduced Planck constant. Substituting Eq. (2) into Eq. (4), we obtain:

$$\frac{d^2 \psi_{nl}(r)}{dr^2} + \left[\zeta + \frac{\alpha_0}{r} - \alpha_1 r + \alpha_2 r^2 - \frac{\gamma}{r^2} \right] \psi_{nl}(r) = 0 \quad (5)$$

where,

$$\left. \begin{aligned} \zeta &= \frac{2\mu}{\hbar^2} (E_{nl} - \beta_d), \quad \alpha_0 = \frac{2\mu\beta_a}{\hbar^2} \\ \alpha_1 &= \frac{2\mu\beta_b}{\hbar^2}, \quad \alpha_2 = \frac{2\mu\beta_c}{\hbar^2}, \quad \gamma = l(l+1) \end{aligned} \right\} \quad (6)$$

By performing a coordinate transformation in Eq. (5), we set $x = \frac{1}{r}$. Differentiating accordingly, simplifying, and substituting back into Eq. (5) yields

$$\frac{d^2\psi_{nl}(x)}{dx^2} + \frac{2}{x} \frac{d\psi_{nl}}{dx} + \frac{1}{x^4} \left[\zeta + \alpha_0 x - \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} - \gamma x^2 \right] \psi_{nl}(x) = 0 \quad (7)$$

We now introduce an appropriate approximation scheme for the terms $\frac{\alpha_1}{x}$ and $\frac{\alpha_2}{x^2}$, in order to facilitate analytical treatment of the equation. Let us assume that the meson possesses a characteristic radius r_0 . Based on this, we employ an approximation scheme by expanding the relevant terms in a power series about r_0 , i.e., around the point $\delta_0 \equiv \frac{1}{r_0}$, up to second order. At this stage, the series expansion is carried out around a

characteristic radius r_0 , which corresponds to the most probable interquark separation in the bound state. For heavy quarkonium systems, the large quark masses and the confining interaction lead to a strong spatial localization of the radial wavefunctions around r_0 . As a result, the dominant contributions to bound-state observables arise from this region. Consequently, it remains reliable over the physically relevant domain of the bound state. This procedure follows the Pekeris approximation, which deforms the centrifugal term into a solvable form and enables the application of the Nikiforov–Uvarov method [21]. In the present calculation, the same value of ($\delta_0=0.1000$) was used for both charmonium and bottomonium, as indicated in Table 1, to maintain consistency in the Pekeris approximation.

By setting $y = x - \delta_0$ and expanding around $y = 0$, the function can be expressed as a power series in x , given by:

$$\frac{\alpha_1}{x} = \alpha_1 \left(\frac{3}{\delta_0} - \frac{3x}{\delta_0^2} + \frac{x^2}{\delta_0^3} \right) \quad (8)$$

Analogously,

$$\frac{\alpha_2}{x^2} = \alpha_2 \left(\frac{6}{\delta_0^2} - \frac{8x}{\delta_0^3} + \frac{3x^2}{\delta_0^4} \right) \quad (9)$$

Inserting Eqs. (8) and (9) into Eq. (7) yields:

$$\frac{d^2\psi(x)}{dx^2} + \frac{2x}{x^2} \frac{d\psi(x)}{dx} + \frac{1}{x^4} [-\varepsilon + \alpha_3 x - \beta x^2] \psi(x) = 0 \quad (10)$$

Such that

$$-\varepsilon = \left(\zeta + \frac{6\alpha_2}{\delta_0^2} - \frac{3\alpha_1}{\delta_0} \right), \quad \alpha_3 = \left(\frac{3\alpha_1}{\delta_0^2} + \alpha_0 - \frac{8\alpha_2}{\delta_0^3} \right), \quad \beta = \left(\gamma + \frac{\alpha_1}{\delta_0^3} - \frac{3\alpha_2}{\delta_0^4} \right) \quad (11)$$

A comparison between Eq. (10) and Eq. (A1) gives

$$\left. \begin{aligned} \tilde{\tau}(x) &= 2x, \quad \sigma(x) = x^2 \\ \tilde{\sigma}(x) &= -\varepsilon + \alpha_3 x - \beta x^2 \\ \sigma'(x) &= 2x, \quad \sigma''(x) = 2 \end{aligned} \right\} \quad (12)$$

Inserting Eq. (12) into Eq. (A9) yields

$$\pi(x) = \pm \sqrt{\varepsilon - \alpha_3 x + (\beta + k)x^2} \quad (13)$$

The value of k is obtained by computing the discriminant of the radicand, resulting in

$$k = \frac{\alpha_3^2 - 4\beta\varepsilon}{4\varepsilon} \quad (14)$$

Inserting Eq. (14) into Eq. (13) yields:

$$\pi(x) = \pm \left(\frac{\alpha_3 x}{2\sqrt{\varepsilon}} - \frac{\varepsilon}{\sqrt{\varepsilon}} \right) \quad (15)$$

The negative branch is selected to satisfy the NU bound-state condition $\tau'(x) < 0$ and to ensure physically acceptable, normalizable solutions.

Differentiation then gives:

$$\pi'_-(x) = -\frac{\alpha_3}{2\sqrt{\varepsilon}} \quad (16)$$

Inserting Eqs. (12) and (16) into Eq. (A7) yields:

$$\tau(x) = 2x - \frac{\alpha_3 x}{\sqrt{\varepsilon}} + \frac{2\varepsilon}{\sqrt{\varepsilon}} \quad (17)$$

Taking the derivative of Eq. (17) yields

$$\tau'(x) = 2 - \frac{\alpha_3}{\sqrt{\varepsilon}} \quad (18)$$

With the aid of Eq. (A10), we arrive at:

$$\lambda = \frac{\alpha_3^2 - 4\beta\varepsilon}{4\varepsilon} - \frac{\alpha_3}{2\sqrt{\varepsilon}} \quad (19)$$

By proceeding with Eq. (A11), we obtain the following result

$$\lambda_n = \frac{n\alpha_3}{\sqrt{\varepsilon}} - n^2 - n \quad (20)$$

Upon equating Eqs. (19) and (20), and using Eqs. (3) and (6), the expression for the energy eigenvalue becomes Eq. (21). The total radial wavefunction is written as $\psi(x) = \phi(x)y(x)$, where $\phi(x)$ is obtained from the logarithmic derivative relation in Eq. (A4), and the polynomial solutions $y(x)$ are generated using the Rodrigues formula in Eq. (A5), leading to the normalized wavefunction given in Eq. (22).

$$E_{nl} = A_0 + \alpha A_0 B_0 - \alpha D_0 + \frac{A_0 B_0 \alpha^3}{\delta_0^2} - \frac{D_0 \alpha^3}{\delta_0^2} - \frac{3A_0 B_0 \alpha^2}{2\delta_0} + \frac{3D_0 \alpha^2}{2\delta_0} - \frac{\hbar^2}{8\mu} \left[\frac{\frac{2\mu}{\hbar^2}(A_0 B_0 + C_0 - D_0) + \frac{3\mu}{\hbar^2 \delta_0^2}(-A_0 B_0 \alpha^2 + \alpha^2 D_0) - \frac{16\mu}{6\hbar^2 \delta_0^3}(-A_0 B_0 \alpha^3 + D_0 \alpha^3)}{n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{\mu}{\hbar^2 \delta_0^3}(-A_0 B_0 \alpha^2 + \alpha^2 D_0) - \frac{\mu}{\hbar^2 \delta_0^4}(-A_0 B_0 \alpha^3 + \alpha^3 D_0)}} \right]^2 \quad (21)$$

The solution for the wavefunction, involving Laguerre polynomials, takes the form

$$\psi_{nl}(x) = N_{nl} x^{-\frac{\alpha}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{x\sqrt{\varepsilon}}} L_n^{\frac{\alpha}{\sqrt{\varepsilon}}} \left(\frac{2\varepsilon}{x\sqrt{\varepsilon}} \right) \quad (22)$$

The constant N_{nl} is the normalization factor, derived using the following condition

$$\int_0^\infty \psi_{nl}(r) \psi_{nl}^*(r) dr = 1 \quad (23)$$

To evaluate the mass spectra of heavy quarkonium systems (e.g., charmonium and bottomonium), composed of quark–antiquark pairs of the same flavor, we employ the following relation [22]:

$$M = 2m + E_{nl} \quad (24)$$

In this framework, m denotes the quark mass, while E_{nl} represents the corresponding energy eigenvalues. Substituting Eq. (21) into Eq. (24), we obtain the expressions for the mass spectra of the VHP:

$$M = 2m + A_0 + \alpha A_0 B_0 - \alpha D_0 + \frac{A_0 B_0 \alpha^3}{\delta_0^2} - \frac{D_0 \alpha^3}{\delta_0^2} - \frac{3A_0 B_0 \alpha^2}{2\delta_0} + \frac{3D_0 \alpha^2}{2\delta_0} - \frac{\hbar^2}{8\mu} \left[\frac{\frac{2\mu}{\hbar^2}(A_0 B_0 + C_0 - D_0) + \frac{3\mu}{\hbar^2 \delta_0^2}(-A_0 B_0 \alpha^2 + \alpha^2 D_0) - \frac{16\mu}{6\hbar^2 \delta_0^3}(-A_0 B_0 \alpha^3 + D_0 \alpha^3)}{n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 + \frac{\mu}{\hbar^2 \delta_0^3}(-A_0 B_0 \alpha^2 + \alpha^2 D_0) - \frac{\mu}{\hbar^2 \delta_0^4}(-A_0 B_0 \alpha^3 + \alpha^3 D_0)}} \right]^2 \quad (25)$$

3. Decay Constants and Transition Rates

The predictive power of a potential model for heavy quarkonia extends beyond the mass spectrum to observables such as leptonic widths and vector decay constants. These quantities depend on the short-distance behaviour of the quarkonium wavefunction and, for an S-wave state, specifically on the finite radial wavefunction at the origin, $\langle R(0) \rangle$. It is therefore important to distinguish between the reduced radial function and the physical radial function. These observables are sensitive to the wavefunction at the origin, $|\psi_{n0}(0)|^2$, which encapsulates the short-distance behavior of the quark–antiquark system [23]. From Eq. (22), the radial wavefunction can be further expressed as

$$\psi_{nl}(r) = N_{nl} r^s e^{-k_1 r} L_n^{(2s)}(2k_1 r) \quad (26)$$

where $L_n^{(2s)}$ are the associated Laguerre polynomials, s and k_1 are the parameters determined by the NU procedure and N_{nl} is the normalization constant defined in Eq. (23)

For S-wave states ($\ell=0$), the radial wavefunction obtained in Eq. (26) becomes

$$\psi_{n,0}(r) = N_{n,0} r^s e^{-k_1 r} L_n^{(2s)}(2k_1 r) \quad (27)$$

Evaluating this at the origin gives

$$\psi_{n,0}(0) = \lim_{r \rightarrow 0} N_{n,0} r^s e^{-k_1 r} L_n^{(2s)}(2k_1 r) \quad (28)$$

Since $e^{-k_1 r} \rightarrow 1$ as $r \rightarrow 0$ and the associated Laguerre polynomial satisfies

$$L_n^{(\Omega)}(0) = \frac{\Gamma(n + \Omega + 1)}{\Gamma(\Omega + 1)n!} \quad (29)$$

Putting $\Omega = 2s$ into Eq. (29) gives

$$L_n^{(2s)}(0) = \frac{\Gamma(n + 2s + 1)}{\Gamma(2s + 1)n!} \quad (30)$$

$$\text{Hence, } \psi_{n,0}(0) = N_{n,0} \frac{\Gamma(n + 2s + 1)}{\Gamma(2s + 1)n!} \lim_{r \rightarrow 0} r^s \quad (31)$$

The probability density at the origin is given as

$$|\psi_{n,0}(0)|^2 = |N_{n,0}|^2 \left(\frac{\Gamma(n + 2s + 1)}{\Gamma(2s + 1)n!} \right)^2 \lim_{r \rightarrow 0} r^{2s} \quad (32)$$

3.1 Leptonic widths

For heavy quarkonium systems, leptonic decay widths are governed by the short-distance behavior of the quark–antiquark wavefunction. The total wavefunction is written as $\psi(r) = R(r)Y_{lm}(\theta, \varphi)$, where the radial function satisfies $R(r) = \chi(r)/r$ with $\chi(r)$ denoting the reduced radial wavefunction. For S-wave states ($\ell =$

0), the spherical harmonic reduces to $Y_{00} = 1/\sqrt{4\pi}$, and therefore the wavefunction at the origin is given by $\psi(0) = \frac{\chi(r)}{r} \Big|_{r=0} \frac{1}{\sqrt{4\pi}}$. Consequently, the physically relevant quantity entering leptonic decay widths is $|\psi(0)|^2 = |\chi(0)|^2 / (4\pi)$. Accordingly, the Van Royen–Weisskopf formula for the leptonic decay width of a vector meson, including the leading-order QCD radiative correction, is written as [24].

$$\Gamma_{ee}(V \rightarrow e^+e^-) = \frac{16\pi\alpha_f^2 e_Q^2}{M_V^2} |\psi(0)|^2 \left(1 - \frac{16\alpha_s}{3\pi}\right), \quad (33)$$

where $\alpha_f \simeq 1/137.036$ is the fine-structure constant, e_Q is the heavy-quark charge, M_V is the vector meson mass, and α_s is the QCD running coupling.

3.2 Vector decay constants

The leptonic decay constant f_V is defined through the matrix element of the vector current and is related to the leptonic width by [24]:

$$\Gamma_{ee}(V \rightarrow e^+e^-) = \frac{4\pi\alpha_f^2 e_Q^2}{3M_V} f_V^2 \left[1 - (16/3\pi)\alpha_s\right] \quad (34)$$

leading to

$$f_V^2 = \left(\frac{12}{M_V}\right) |\psi(0)|^2, \quad (35)$$

Equations (33) and (35) together allow the extraction of decay constants directly from the normalized wavefunctions obtained with the Varshni–Hellmann potential. Experimental reference values for leptonic widths are available in the Particle Data Group listings [26], enabling direct comparison with theoretical predictions.

The mean absolute error (MAE) is given as

$$\frac{1}{N} \sum |M_i^T - M_i^E| \quad (36)$$

The variables appearing in this expression are defined as follows: N denotes the number of available experimental data points, ME represents the experimental values, and M^T corresponds to the theoretically calculated values.

4. Discussion of results

The mass spectra of charmonium and bottomonium states, from 1S to 1F, were computed using Eq. (25). The potential parameters were determined simultaneously from the experimentally established (1S), (2S), (3S), and (4S) mass levels for each quarkonium system, subject to the physical constraints given in Table 1. Reference values were taken from Refs. [27,28].

Table 1. Parameters of the Varshni–Hellmann potential and the corresponding properties of the heavy mesons.

Parameters	Charmonium	Bottomonium
A_0	1.3400 GeV	1.6100 GeV
B_0	4.9675 GeV ⁻¹	4.6513 GeV ⁻¹
C_0	0.9217	5.4212
D_0	6.6565	7.4885
α	0.0100 GeV	0.0100 GeV
δ_0	0.1000	0.1000
μ	0.6045 GeV	2.4115 GeV

Constituent quark masses of 1.209 GeV for charmonium and 4.823 GeV for bottomonium yield reduced masses of 0.6045 GeV and 2.4115 GeV, respectively [29]. The results, summarized in Tables 2 and 3, demonstrate the effectiveness of the Varshni–Hellmann potential, which combines a short-range Coulomb-like term with a long-range screened interaction, capturing essential QCD features such as asymptotic freedom at short distances and confinement at intermediate ranges. For bottomonium, the calculated masses show an agreement with experimental data across all states, from the tightly bound 1S to higher excitations such as 4S and 1D. The model reproduces fine spectral structures arising from the strong binding of the heavy quark, which leads to compressed spatial configurations, sharper wavefunctions, and wider level spacings, characteristic features of the heavy-quark regime in QCD.

Table 2. Mass spectra of bottomonium (GeV) for Varshni–Hellmann potential

State	Present work (PW)	[22]	[18]	Experiment [27,28]	Error (PW)	Error [22]	Error [18]
1S	9.460	9.460	9.462	9.460	0.000	0.000	0.002
2S	10.023	10.023	10.027	10.023	0.000	0.000	0.004
1P	9.899	9.619	9.963	9.899	0.000	0.280	0.064
2P	10.260	10.114	10.299	10.260	0.000	0.146	0.039
3S	10.356	10.355	10.361	10.355	0.001	0.000	0.006
4S	10.580	10.567	10.624	10.580	0.000	0.013	0.044
1D	10.165	9.864	10.209	10.164	0.001	0.300	0.045
2D	10.307	-	-	-	-	-	-
1F	10.210	-	-	-	-	-	-

Note: “—” indicates that no corresponding experimental or reference value is available; the present-work value is therefore a model prediction.

Table 3. Mass spectra of charmonium (GeV) for Varshni–Hellmann potential

State	Present work (PW)	[22]	[18]	Experiment [27,28]	Error (PW)	Error [22]	Error [18]
1S	3.096	3.096	3.096	3.096	0.000	0.000	0.000
2S	3.686	3.686	3.672	3.686	0.000	0.000	0.014
1P	3.525	3.255	3.521	3.525	0.000	0.270	0.004
2P	3.773	3.779	3.951	3.773	0.000	0.006	0.178
3S	4.040	4.040	4.085	4.040	0.000	0.000	0.045
4S	4.264	4.269	4.433	4.263	0.001	0.006	0.170
1D	3.771	3.504	3.800	3.770	0.001	0.266	0.030
2D	4.160	-	-	4.159	0.001	-	-
1F	3.962	-	-	-	-	-	-

Note: “—” indicates that no corresponding experimental or reference value is available; the present-work value is therefore a model prediction.

For bottomonium, the present work (PW) shows good agreement with the experimental value, with a sum of absolute deviations (SAD) of 0.002 GeV and a mean absolute error (MAE) of 0.00029 GeV, compared with 0.204 GeV and 0.02914 GeV, respectively, for Ref. [18], and 0.739 GeV and 0.10557 GeV for Ref. [22]. Similarly, for charmonium, PW gives a sum of absolute deviations (SAD) of 0.003 GeV and a mean absolute error (MAE) of 0.00038 GeV, substantially lower than those of Ref. [18] (0.441 GeV, 0.06300 GeV) and Ref. [22] (0.548 GeV, 0.07829 GeV). These results demonstrate the accuracy of the Varshni–Hellmann potential in reproducing the observed heavy-quarkonium mass spectra. Using established values of the wavefunction at the origin $|R_{J/\psi}(0)|^2 = 0.81 \text{ GeV}^3$ and $|R_{Y(1S)}(0)|^2 = 6.48 \text{ GeV}^3$, $\alpha_s(m_c) = 0.26$, $\alpha_s(m_b) = 0.18$ together with the Van Royen–Weisskopf formula and the one-loop QCD correction in Eq. (33), the leptonic widths and vector decay constants listed in Table 4 were evaluated. The results in Table 4 show that the calculated leptonic widths provide reasonable, although not exact, agreement with the experimental values. For $J/\psi(1S)$, the calculated width of 4.467 keV differs from the experimental value of (5.55 ± 0.145) keV by approximately 20%. For $Y(1S)$, the calculated width of 1.190 keV differs from the experimental value of (1.340 ± 0.018) keV by approximately 11%.

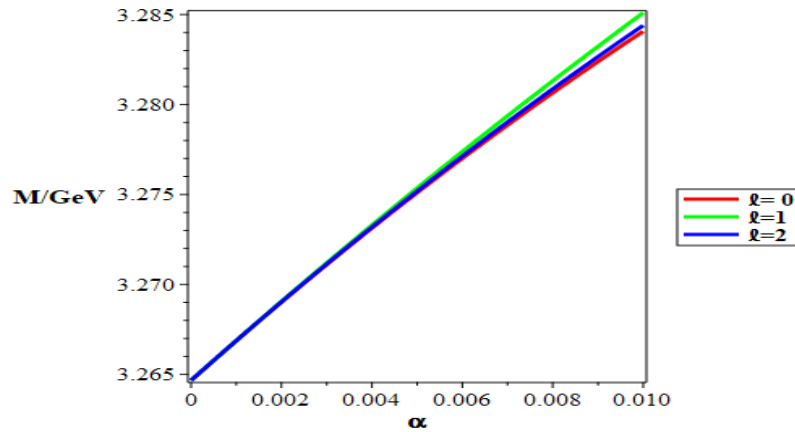


Fig.1 Mass spectra as a function of the screening parameter for different angular quantum numbers

These deviations indicate that, although the present non-relativistic treatment captures the dominant short-distance contribution, the decay observables remain sensitive to effects not explicitly included in the present model, including higher-order QCD radiative corrections, relativistic corrections, and uncertainties associated with the chosen strong-coupling scale. Therefore, the decay-width results should be regarded as an additional validation of the model rather than as a precision determination of the leptonic widths.

Table 4: Leptonic widths (Γ_{ee}) and vector decay constants (f_V) for S-wave heavy mesons.

State	$\Gamma_{ee} (KeV)$	$f_V (MeV)$	PDG $\Gamma_{ee} (KeV)$ [26]
$J/\psi(1S)$	4.467	374	5.550 ± 0.140
$\Upsilon(1S)$	1.190	674	1.340 ± 0.018

Figure 1 illustrates the dependence of the calculated mass spectrum on the screening parameter (α) for orbital angular quantum numbers $\ell = 0, 1$, and 2. The parameter (α) controls the exponential screening of the long-range component of the Varshni–Hellmann interaction and therefore determines the effective range over which the corresponding attractive interaction remains significant. As (α) increases, the long-range interaction becomes more strongly screened, modifying the effective binding between the quark and antiquark and producing a systematic shift in the energy eigenvalues and meson masses. The approximately linear increase in the calculated masses with (α) over the range considered indicates that screening produces a smooth modification of the quark–antiquark interaction for the parameter values investigated. States with higher angular momentum are additionally affected by the centrifugal contribution, which leads to differences in their spectral response to changes in (α). The observed dependence therefore reflects the combined effects of short-distance binding, screened long-range interaction, and the centrifugal contribution.

Wavefunction profiles in Figs. 2 and 3 reveal further structural features. For charmonium (Fig. 2), the wavefunctions exhibit radial nodes and spatial broadening with increasing excitation, consistent with weaker binding. In contrast, bottomonium wavefunctions (Fig. 3) remain sharply localized near the origin, reflecting the strong binding and compact size characteristic of systems dominated by a heavy quark mass. These results demonstrate the robustness of the Varshni–Hellmann potential as a reliable and analytically tractable framework for modeling heavy quark–antiquark systems. Its success in reproducing experimental spectra and capturing key spatial and dynamical features highlights its value for studying not only charmonium and bottomonium but also exotic hadronic states and quarkonium behavior under extreme QCD conditions such as those probed in heavy-ion collisions.

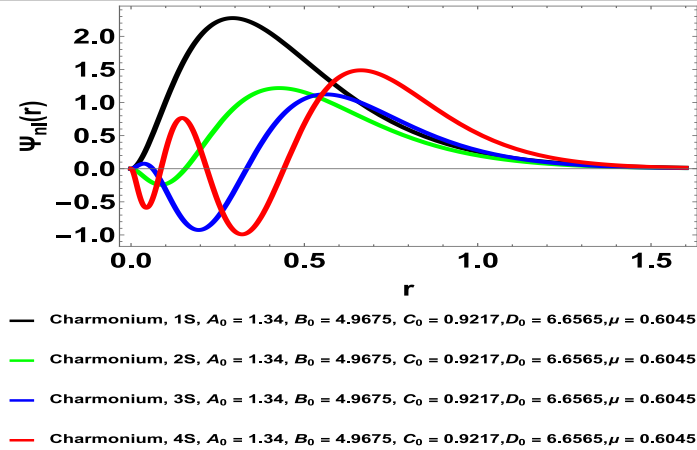


Fig.2. Wavefunctions of charmonium for different S states.

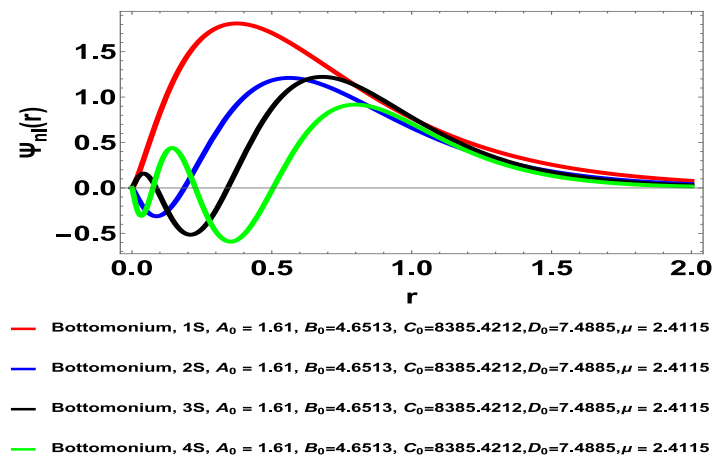


Fig.3 Wavefunctions of bottomonium for different S states.

4. Conclusion

In this study, the Varshni–Hellmann potential was employed to investigate the spectroscopy and structural properties of heavy quark–antiquark systems. The Schrödinger equation was solved analytically using the Nikiforov–Uvarov method, yielding energy spectra and corresponding wavefunctions for different orbital states. The model reproduces the experimental mass spectra with high accuracy, with a sum of absolute deviations (SAD) of 0.002 GeV and a mean absolute error (MAE) of 0.00029 GeV for bottomonium, and SAD of 0.003 GeV and MAE of 0.00038 GeV for charmonium. This close agreement demonstrates the effectiveness of the combined Coulomb-like short-range and screened long-range components of the potential, together with the fitted model parameters.

The calculated leptonic widths and decay constants of the S-wave states, obtained within the Van Royen–Weisskopf framework, show reasonable agreement with the Particle Data Group values and results from phenomenological and lattice QCD studies. Predictions for the P- and D-wave states provide an additional assessment of the model beyond the S-wave states used for parameter determination. Overall, the Varshni–Hellmann potential provides an analytically tractable and effective framework for describing heavy quarkonium spectra and related decay observables. Future studies may extend this framework to exotic hadronic systems and medium-dependent quarkonium properties in extreme QCD environments, including screening and dissociation phenomena.

Conflict of interest statement

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

CRedit author statement

Inyang E. P.: Conceptualization, Methodology, Investigation; Writing – original draft; **Valentine, N. H. and Osang J.E.:** Data Curation, Writing Original Draft; **William E.S. and Mopta S.E:** Methodology, Resources, Validation; **Amajama J. and Ekechukwu C.C.:** Resources. The final manuscript was read and approved by all authors.

Statement on the use of Artificial Intelligence.

During the preparation of this manuscript, artificial intelligence tools were used solely for language editing and grammatical improvement. No AI tools were used to generate scientific content, analysis, results, or conclusions.

Data Availability Statement

No external datasets were used in this study. All data were generated numerically from the analytical solutions.

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References

1. Litvinov, A., & Meshcheriakov, P. (2025). Meson mass spectrum in QCD2 't Hooft's model. *Nuclear Physics B*, 1010, 116766. <https://doi.org/10.1016/j.nuclphysb.2025.116766>
2. Inyang, E. P., Ali, N., Endut, R., Rusli, N., & Aljunid S.A. (2025). The radial scalar power potential and its application to quarkonium systems. *Indian Journal of Physics*, 99, 715–724. <https://doi.org/10.1007/s12648-024-03335-9>
3. Ali, M. S., Hassan, G. S., Abdelmonem, A. M., Elshamdy, S. K., Elmasry, F., & Yasser, A. M. (2020). The spectrum of charmed quarkonium in non-relativistic quark model using matrix Numerov's method. *Journal of Radiation Research and Applied Sciences*, 13, 226–233. <https://doi.org/10.1080/16878507.2020.1723949>
4. Inyang, E. P., Ali, N., Endut, R., Rusli, N., Aljunid, S. A., Ali, N. R., & Asjad M. M. (2024). Thermal properties and mass spectra of heavy mesons in the presence of a point-like defect. *East European Journal of Physics*, 1, 156–166. <https://doi.org/10.26565/2312-4334-2024-1-14>
5. Matvienko, D. (2018). The Belle II experiment: Status and physics program. *EPJ Web of Conferences*, 191, 02010. <https://doi.org/10.1051/epjconf/201819102010>
6. Mistry, R., Shah, M., & Majethiya, A. (2024). Mass spectra, Regge trajectories and decay properties of heavy-flavour mesons. *Revista Mexicana de Física*, 70, 010801. <https://doi.org/10.31349/RevMexFis.70.010801>
7. Inyang, E. P., Nwachukwu, I. M., Ekechukwu, C. C., Ekong, I. B., William, E. S., Lawal, K. M., Momoh, K. O., & Oyelami, O. A. (2024). Analytical solution of the class of inversely quadratic Yukawa potential with application to quantum mechanical systems. *Eurasian Physical Technical Journal*, 21(4), 118–130. <https://doi.org/10.31489/2024No4/118-130>
8. Varshni, Y. P. (1957). Comparative study of potential energy functions for diatomic molecules. *Reviews of Modern Physics*, 29, 664–669. <https://doi.org/10.1103/RevModPhys.29.664>
9. Hellmann, H. (1935). A new approximation method in the problem of many electrons. *The Journal of Chemical Physics*, 3, 61. <https://doi.org/10.1063/1.1749559>
10. Purohit, K. R., Rai, A. K., & Parmar, R. H. (2024). Spectroscopy of heavy-light mesons (D, Ds, B, Bs) for the linear plus modified Yukawa potential using Nikiforov–Uvarov method. *Indian Journal of Physics*, 98, 1109–1121. <https://doi.org/10.1007/s12648-023-02852-3>
11. Rai, A. K., & Rathaud, D. P. (2015). The mass spectra and decay properties of dimesonic states, using the Hellmann potential. *European Physical Journal C*, 75, 462. <https://doi.org/10.1140/epjc/s10052-015-3695-z>
12. Inyang, E. P., Inyang, E. P., William, E. S., & Ibekwe, E. E. (2021). Study on the applicability of Varshni potential to predict the mass spectra of the quark–antiquark systems in a non-relativistic framework. *Jordan Journal of Physics*, 14(4). <https://doi.org/10.47011/14.4.8>
13. Inyang, E. P., Obisung, E. O., Iwuji, P. C., Ntibi, J. E., Amajama, J., & William, E. S. (2022). Masses and thermal properties of a charmonium and bottomonium mesons. *Journal of the Nigerian Society of Physical Sciences*, 4(3), 884. <https://doi.org/10.46481/jnsps.2022.884>

14. Yin, P. L., Chen, C., Krein, G., Roberts, C. D., Segovia, J., & Xu, S. S. (2019). Masses of ground-state mesons and baryons, including those with heavy quarks. *Physical Review D*, 100, 034008. <https://doi.org/10.1103/PhysRevD.100.034008>
15. Ibekwe, E. E., Emah, J. B., Inyang, E. P., & Akpan, A. O. (2022). Mass spectrum of heavy quarkonium for combined potentials (modified Kratzer plus screened Coulomb potential). *Iranian Journal of Science and Technology*, 46, 1748. <https://doi.org/10.1007/s40995-022-01264-2>
16. Abu-Shady, M., & Fath-Allah, H. M. (2025). Thermodynamic analysis and mass spectra of heavy mesons via the generalized fractional Klein–Gordon equation. *Revista Mexicana de Física*, 71, 030801. <https://doi.org/10.31349/RevMexFis.71.030801>
17. Abu-Shady, M., & Fath-Allah, H. M. (2025). Investigating heavy quarkonia binding in an anisotropic-dense quark–gluon plasma with topological defects in the framework of fractional non-relativistic quark model. *Scientific Reports*, 15, 1875. <https://doi.org/10.1038/s41598-025-01875-2>
18. Ciftci, H., & Kisoglu, H. F. (2018). Non-relativistic arbitrary l-states of quarkonium through asymptotic iteration method. *Advances in High Energy Physics*, 2018, 4549705. <https://doi.org/10.1155/2018/4549705>
19. William, E., Inyang, E., Ntibi, J. E., Obu, J. A., & Inyang, E. P. (2022). Solutions of the non-relativistic equation interacting with the Varshni–Hellmann potential model with some selected diatomic molecules. *Jordan Journal of Physics*, 15, 179–193. <https://doi.org/10.47011/15.2.9>
20. Inyang, E. P., Aouami, A. E. L., Ali, N., Endut, R., Ali, N. R., & Aljunid, S. A. (2024). Information entropies with Varshni–Hellmann potential in higher dimensions. *Physics Open*, 20, 100220. <https://doi.org/10.1016/j.physo.2024.100220>
21. Nikiforov, S. K., & Uvarov, V. B. (1988). *Special functions of mathematical physics*. Birkhäuser, Basel. <https://doi.org/10.1007/978-1-4757-1595-8>
22. Abu-Shady, M., Abdel-Karim, T. A., & Ezz-Alarab, Y. (2019). Masses and thermodynamic properties of heavy mesons in the non-relativistic quark model using the Nikiforov–Uvarov method. *Journal of the Egyptian Mathematical Society*, 27, 145. <https://doi.org/10.1186/s42787-019-0022-2>
23. Van Royen, R., & Weisskopf, V.F. (1967). Hadron decay processes and the quark model. *Il Nuovo Cimento A* (1965–1970), 50(3), 617–645. <https://doi.org/10.1007/BF02823542>
24. Olavo, L. S. F. (1999). Foundations of quantum mechanics: Non-relativistic theory. *Physica A: Statistical Mechanics and its Applications*, 262(1–2), 197–214. [https://doi.org/10.1016/S0378-4371\(98\)00395-1](https://doi.org/10.1016/S0378-4371(98)00395-1)
25. Wittig, H. (2020). QCD on the lattice. In *Particle Physics Reference Library, Vol. 1: Theory and Experiments* (pp. 137–262). Springer, Cham. https://doi.org/10.1007/978-3-030-38207-0_5
26. Navas, S., Amsler, C., Gutsche, T., Hanhart, C., Hernández-Rey, J. J., Lourenço, C., Masoni, A., Mikhasenko, M., Mitchell, R. E., & Particle Data Group. (2024). Review of particle physics. *Physical Review D*, 110(3), 030001. <https://doi.org/10.1103/PhysRevD.110.030001>
27. Tanabashi, M., Carone, C. D., Trippie, T. G., Wohl, C. G., & Particle Data Group. (2018). Review of particle physics. *Physical Review D*, 98, 030001. <https://doi.org/10.1103/PhysRevD.98.030001>
28. Olive, K. A., Groom, D. E., Trippie, T. G., & Particle Data Group. (2014). Review of particle physics. *Chinese Physics C*, 38, 090001. <https://doi.org/10.1088/1674-1137/38/9/090001>
29. Barnett, R. M., Carone, C. D., Groom, D. E., Trippie, T. G., Wohl, C. G., & Particle Data Group. (2012). Review of particle physics. *Physical Review D*, 86, 010001. <https://doi.org/10.1103/PhysRevD.86.010001>

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APPENDIX A: Review of Nikiforov-Uvarov (NU) method

The NU method was proposed by Nikiforov and Uvarov [33] to transform Schrödinger-like equations into a second-order differential equation via a coordinate transformation $x = x(r)$, of the form

$$\psi''(x) + \frac{\tilde{\tau}(x)}{\sigma(x)}\psi'(x) + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}\psi(x) = 0 \quad (\text{A1})$$

where $\tilde{\sigma}(x)$, and $\sigma(x)$ are polynomials, at most second degree and $\tilde{\tau}(x)$ is a first-degree polynomial. The exact solution of Eq.(A1) can be obtain by using the transformation.

$$\psi(x) = \phi(x)y(x) \quad (\text{A2})$$

This transformation reduces Eq.(A1) into a hypergeometric-type equation of the form

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0 \quad (\text{A3})$$

The function $\phi(x)$ can be defined as the logarithm derivative

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)} \quad (\text{A4})$$

With $\pi(x)$ being at most a first-degree polynomial. The second part of $\psi(x)$ being $y(x)$ in Eq.(A2) is the hypergeometric function with its polynomial solution given by Rodrigues relation as

$$y(x) = \frac{B_{nl}}{\rho(x)} \frac{d^n}{dx^n} [\sigma^n(x)\rho(x)] \quad (\text{A5})$$

Where B_{nl} is the normalization constant and $\rho(x)$ the weight function which satisfies the condition below;

$$(\sigma(x)\rho(x))' = \tau(x)\rho(x) \quad (\text{A6})$$

where, also

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x) \quad (\text{A7})$$

For bound solutions, it is required that

$$\tau'(x) < 0 \quad (\text{A8})$$

The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(x)$ and parameter λ , respectively:

$$\pi(x) = \frac{\sigma'(x) - \tilde{\tau}(x)}{2} \pm \sqrt{\left(\frac{\sigma'(x) - \tilde{\tau}(x)}{2}\right)^2 - \tilde{\sigma}(x) + k\sigma(x)} \quad (\text{A9})$$

And

$$\lambda = k_- + \pi'_-(x) \quad (\text{A10})$$

The value of k can be obtained by setting the discriminant in the square root in Eq. (A9) equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda + n\tau'(x) + \frac{n(n-1)}{2}\sigma''(x) = 0, (n = 0, 1, 2, \dots) \quad (\text{A11})$$