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PARAMETRIZED EIGHT-VERTEX MODEL AND KNOT INVARIANT 10_{136}

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The article discusses and expands the known elements of the eight-vertex model, paying special attention to the parameterization of the matrix. The matrix values are interconnected with the knot through the braids and this model is valid on finite square lattices in two-dimensional space. A new solution of the parametrized eight-vertex model of free fermions with a complex version of elliptic functions, which is valid on a finite lattice, will be constructed. The range of applicability of the eight-vertex model with elements of the Jacobi elliptic function and the construction of a knot invariant on its basis is discussed by comparing the results obtained analytically for the model. The construction of the knot invariant using the Clebsch-Gordan coefficients and the main tool of statistical mechanics of the Yang-Baxter equation will be studied in detail.

Keywords: 10_{136} knot , elliptic function, Clebsch-Gordan coefficients.

Introduction

In recent years, scientists in the fields of mathematics and physics have been diligently pursuing related theories, especially the theory of knots, which has generated interest for a small number of scientists and has since become one of the most fashionable hobbies of mathematicians, physicists and even geneticists. This, in turn, gave rise to a number of methods linking two, at first glance, very distant from each other areas of mathematics and physics: knot theory and statistical mechanics.

In [1], the Boltzmann weights of the doubled Ising model, parametrized using elliptic functions, are considered, and an alternative way of checking the Yang-Baxter equation in the parameterization of matrix elements using the star-triangle equation is shown. Thus, in [2], a quantum decomposition of the eight-vertex model was introduced and a set of closure properties in various regions of the parameter space was proved. In [3], it was proved that the sixteen-vertex model gives an exact description of the thermodynamics of artificial spin ice models. The work [4] confirms the complexity dichotomy theorem for the eight-vertex model. For each setting of the model parameters, the calculation of the partition function is proved, which is either solvable in polynomial time or is $\#P$ - complexity.

The paper investigates the solution of models of statistical mechanics and knot theory, which in turn, this connection is occupied by the Temperley-Lieb algebra [5] and the Birman-Murakami-Wenzl algebra (BMW) [6]. Also, in [7], the latest research on the connection of vertex models of statistical mechanics with the main problems of mathematics is studied. The origins of the connection with physics go back to the very close relationship between the state models of knot polynomials and the partition function in statistical mechanics. This connection in [8] led to the construction of a number of invariants that go beyond the original skein polynomials. Thus, in the source [9], the Jones polynomial in a closed braid is the partition function of the statistical mechanic's model on the braid. The work [10] summarizes procedure outputting braid generator representations from three-partite vertex model. This representation made it possible to study the invariant of a knot with multi-colored links, where the components of the knot have different spins. The formula for the invariant of knot with a multi-colored link is studied from the point of view of the braid generators obtained from the R-matrices of three-partite vertex models. The resulting knot invariant 52 corresponds to the Jones polynomial and HOMFLY-PT. Description of Boltzmann weights and finding SO (N) for any N spin vertex model of algebra in [11] opened up new problems in the field of statistical physics. The representations of the braid group in [12], obtained from rational conformal field theories, can be used to obtain explicit representations of the Temperley-Lieb-Jones algebras. In the source [13], the Yang-Baxter equation provides both an algebraic and a graphical method in knot theory. The method of commuting transfer matrices in [14] for this equation is a generalization possibility for solving the eight-vertex model. The solution to the quantum Yang-Baxter equation [15] is an R-matrix, which corresponds to the transfer R-matrix of the eight-vertex model of statistical mechanics.

The method for solving invariants is associated with multi-colored links [16], with the statistical sum of the Chern-Simons theory [17] and with obtaining the topological solution (2) of the Chern-Simons theory on S^3 [18]. In similar sources, as in [19], the connection between the invariant of knot theory and a new ten-vertex model of statistical mechanics was studied using the transition of a commuting transfer matrix, including Boltzmann weights in the braid matrix. Subsequently, in the source [20], the solutions are described using the Clebsch-Gordan coefficients and the transformation matrix.

Thus, in [21], the exact solution of the classical two-dimensional eight-vertex model is the greatest achievement in the field of exactly solvable models and the contradiction with the hypothesis of universality and independence of critical indicators from interaction details was studied in the early 70s. All these studies have found a place in the connection of the eight-vertex model with elements of elliptic functions and knot theory.

In this paper, we consider obtaining braid generators from an eight-vertex model by parametrizing matrix elements with a suitable normalization factor using the quantum form of the Clebsch-Gordan coefficients, the so-called $3j$ - symbol. The solution is based on the observation that the rows in the transfer matrices commute for a specific parametrization of the four Boltzmann weights. The presented solution will help to understand how to find the connection between the invariants of knots and lattice models of statistical mechanics through the mathematical apparatus of quantum physics.

The work plan is as follows. In Section 2, we consider the construction of braid matrices from the $R(u)$ -matrix of eight-vertex models for the same spins and the derivation of an algebraic formula for the knot invariant. Section 3 generalizes the procedure for defining a new representation of the braid generators from a $R(u)$-matrix associated with a parametrized vertex model with elliptic functions, and proposes a solution to the invariant of a knot with links. Section 4 presents the results of the work.

1. Parameterized 8-vertex model

The parameterized eight-vertex model is a generalization of the six-vertex model, where spins $j_1 = j_2 = \frac{1}{2}$ are placed on four edges that intersect each vertex Boltzmann weights of $(R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2}(u \to 0)$ matrices are associated with each nonzero vertex if only $m_1 + m_2 = n_1 + n_2$, where $m_1, m_2, n_1, n_2 \in \frac{1}{2}, -\frac{1}{2}$. This condition allows 8 nonzero Boltzmann weights, which are called the parameterized eight-vertex model

$$(R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2}(u \to 0) = \begin{bmatrix} (R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2} & \uparrow\uparrow & \downarrow\uparrow & \uparrow\downarrow & \downarrow\downarrow \\ \uparrow\uparrow & snh(\mu-u) & 0 & 0 & ksnh\mu snhu snh(\mu-u) \\ \downarrow\uparrow & 0 & snhu & snh\mu & 0 \\ \uparrow\downarrow & 0 & snh\mu & snhu & 0 \\ \downarrow\downarrow & ksnh\mu snhu snh(\mu-u) & 0 & 0 & snh(\mu-u) \end{bmatrix}. \quad (1)$$

To construct the braid generators b_i, we take the limit $u \to 0$ and divide the elements of the previous matrix by the Boltzmann weight $(R^{\frac{1}{2},\frac{1}{2}})^{\uparrow,\uparrow}_{\uparrow,\uparrow}(u \to 0)$. Subsequently, we will choose an appropriate normalization so that the matrix elements are finite, as shown below

$$\frac{(R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2}(u \to 0)}{(R^{\frac{1}{2},\frac{1}{2}})^{\uparrow,\uparrow}_{\uparrow,\uparrow}(u \to 0)} = \begin{bmatrix} (R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2} & \uparrow\uparrow & \downarrow\uparrow & \uparrow\downarrow & \downarrow\downarrow \\ \uparrow\uparrow & 1 & 0 & 0 & ksnh\mu snhu \\ \downarrow\uparrow & 0 & \frac{snhu}{snh(\mu-u)} & \frac{snh\mu}{snh(\mu-u)} & 0 \\ \uparrow\downarrow & 0 & \frac{snh\mu}{snh(\mu-u)} & \frac{snhu}{snh(\mu-u)} & 0 \\ \downarrow\downarrow & ksnh\mu snhu & 0 & 0 & 1 \end{bmatrix}. \quad (2)$$

The Wess-Zumino conformal field theory implies a compact relation for the braid generators b_i obtained from $(R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2}$ - the matrix of vertex models, as well as from the eigenvalues (λ) of the monodromy matrix in $SU(2)_k$ [22], where the values of the permutation matrix $(\hat{R}^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2} = \frac{1}{\aleph}\hat{P}^{\frac{1}{2},\frac{1}{2}}(R^{j_1,j_2})^{n_1,n_2}_{m_1,m_2}(u \to 0)$, $\aleph = (R^{j_1,j_2})^{j_1,j_2}_{j_1,j_2}(u \to 0)$ are the normalization factor

$$(\hat{R}^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2} = \frac{1}{\aleph}\hat{P}^{\frac{1}{2},\frac{1}{2}}(R^{j_1,j_2})^{n_1,n_2}_{m_1,m_2}(u \to 0) = \frac{1}{\aleph}\hat{P}^{\frac{1}{2},\frac{1}{2}}\sum_{J \in j \otimes j}\begin{pmatrix} j_1 & j_2 & J \\ m_2 & m_1 & M \end{pmatrix}\lambda_J(j_1, j_2)\begin{pmatrix} j_1 & j_2 & J \\ n_1 & n_2 & M \end{pmatrix}, \tag{3}$$

where $M = m_1 + m_2 = n_1 + n_2$ and the elements in brackets $\begin{pmatrix} j_1 & j_2 & J \\ m_2 & m_1 & M \end{pmatrix}$ denote the quantum form of the formula for the Clebsch-Gordan coefficient $(q - CG)$ [23]. Matrix elements can be expanded in this way

$$\frac{(R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2}(u \to 0)}{(R^{\frac{1}{2},\frac{1}{2}})^{\uparrow,\uparrow}_{\uparrow,\uparrow}(u \to 0)} = \begin{bmatrix} (R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2} & \uparrow\uparrow & \downarrow\uparrow & \uparrow\downarrow & \downarrow\downarrow \\ \uparrow\uparrow & 1 & 0 & 0 & ksnh\mu snhu \\ \downarrow\uparrow & 0 & \frac{snh\mu}{snh(\mu - u)} & \frac{snhu}{snh(\mu - u)} & 0 \\ \uparrow\downarrow & 0 & \frac{snhu}{snh(\mu - u)} & \frac{snh\mu}{snh(\mu - u)} & 0 \\ \downarrow\downarrow & ksnh\mu snhu & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

Therefore, it is possible to define the $b_i\left[\frac{1}{2},\frac{1}{2}\right]$ matrix form of the braid generators using the R matrix. Thus, we calculate the knot invariant by the formula

$$\alpha_{\{j K j\}_n}(A) = \left(\tau_j \overline{\tau_j}\right)^{-\frac{n}{2}}\left(\frac{\overline{\tau_j}}{\tau_j}\right)^{\frac{l}{2}} Tr[HA], \tag{5}$$

where l - sum of exponents b_i appearing in the word of braid A. $H = \underbrace{h_j \otimes h_j K h_j}_{r}$, where r

$$h_j = \frac{1}{1 + q + K + q^{2j}} Diag\left[1, q, K, q^{2j}\right], \tag{6}$$

and τ_j, $\overline{\tau_j}$ are

$$\tau_j = \frac{1}{1 + q + K + q^{2j}}, \tau_j = \frac{1}{1 + q + K + q^{2j}}. \tag{7}$$

Initially, there is only one braid generator b_i, for all braids $A \in B_2$, the matrix form of which will have the form 4×4, where the matrix elements are expressed by elliptic functions with a complex version

$$b_i = (R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2}. \tag{8}$$

An invariant in the equation (5) for knot 1 will be developed according to the Rolfsen table. For knots and links obtained by closing the words of braid $A = b^{10}_{136}$, which are matrix 4×4

$$b_1 = (R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2} \times \Im, \tag{9}$$

$$b_2 = \Im \times (R^{\frac{1}{2},\frac{1}{2}})^{n_1,n_2}_{m_1,m_2}. \tag{10}$$

For example, knot 10_{136} is word braid $A = b^{10}_{136}$. It is important that such an action of the word braid on a braid of five strands implies the following order of the matrix operation on the initial state $\left| j, m_1; j_2, m_2 \right\rangle$

$$A\left|4 - strand\right\rangle \equiv b^{10}_{136}\left| j, m_1; j_2, m_2 \right\rangle. \tag{11}$$

From the above it follows that to derive the eigenvalue, depending on the spectral parameters $\lambda_J(j_1, j_2; u)$ for spin $j_1 = j_2 = 1/2$ such, belong to the parametrized eight-vertex model on a square lattice. This assumption $\lambda_J(j_1, j_2; u)$ was formulated in [22]

$$\lambda_J(j_1, j_2; u) = \prod_{k_1=1}^{J} (snh(k_1\mu - u)) \prod_{k_2=J+1}^{2J} (snh(k_2\mu + u)), \tag{12}$$

accordingly, the eigenvalues of the matrix elements are elliptic functions with a complex version

$$\lambda_0(u) = snh(\mu + u),\ \lambda_1(u) = snh(\mu - u). \tag{13}$$

Thus,

$$\lambda_0(0) = \lambda_1(0) = snh\mu \equiv B(\mu),$$

$$\lambda_0(u)\lambda_0(-u) = \lambda_1(u)\lambda_1(-u) = \chi_1(u)\chi_1(-u) + \chi_4(u)\chi_4(-u) = \chi_3(u)\chi_3(-u) + \chi_2(u)\chi_2(-u) =$$
$$= \chi_2(u)\chi_2(-u) + \chi'_3(u)\chi'_3(-u) = snh(\mu + u)snh(\mu - u) \equiv A(\mu; u)$$

as a result, matrices are obtained depending on the spectral parameter, where $(R^{j_1, j_2})^{n_1, n_2}_{m_1, m_2}(u \to 0)$ are matrices associated with new vertex models

$$(R^{j_1, j_2})^{n_1, n_2}_{m_1, m_2}(u) = \sum_{J \in j \otimes j} \begin{pmatrix} j_1 & j_2 & J \\ m_2 & m_1 & M \end{pmatrix} \lambda_J(j_1, j_2) \begin{pmatrix} j_1 & j_2 & J \\ n_1 & n_2 & M \end{pmatrix}, \tag{14}$$

and the general view of the R-matrix will be

$$(R^{\frac{1}{2}, \frac{1}{2}})^{n_1, n_2}_{m_1, m_2}(u) = \begin{bmatrix} m_1, m_2 / n_1, n_2 & \frac{1}{2}, \frac{1}{2} & \frac{1}{2}, -\frac{1}{2} & -\frac{1}{2}, \frac{1}{2} & -\frac{1}{2}, -\frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} & \chi_1(u) & 0 & 0 & \chi_{41}(u) \\ \frac{1}{2}, -\frac{1}{2} & 0 & \chi_2(u) & \chi_3(u) & 0 \\ -\frac{1}{2}, \frac{1}{2} & 0 & \chi_3(u) & \chi_2(u) & 0 \\ -\frac{1}{2}, -\frac{1}{2} & \chi_4(u) & 0 & 0 & \chi_1(u) \end{bmatrix}, \tag{15}$$

where

$$\chi_1 = \frac{-i}{[2]} \left(\frac{sn(i\mu)cn(iu)dn(iu) + sn(iu)cn(i\mu)dn(i\mu)}{1 - msn^2(i\mu)sn^2(iu)} \right),$$

$$\chi_2 = \frac{-i}{[2]} \left(\frac{sn^2(i\mu)cn^2(iu)dn^2(iu) + sn(iu)cn^2(i\mu)dn^2(i\mu)}{1 - msn^2(i\mu)sn^2(iu)} \right),$$

$$\chi_3 = -isn(i\mu)e^{-u},\ \chi_4 = 0,\ \chi'_3 = -isn(i\mu)e^{u}.$$

Replacing $u = 0$ и $q = e^{2\mu}$ для $j_1 = j_2 = \frac{1}{2}$, there is

$$\left.\frac{(R^{\frac{1}{2}, \frac{1}{2}})^{n_1, n_2}_{m_1, m_2}(u \to 0)}{(R^{\frac{1}{2}, \frac{1}{2}})^{\uparrow, \uparrow}_{\uparrow, \uparrow}(u \to 0)}\right) = \begin{bmatrix} m_1, m_2 / n_1, n_2 & \frac{1}{2}, \frac{1}{2} & \frac{1}{2}, -\frac{1}{2} & -\frac{1}{2}, \frac{1}{2} & -\frac{1}{2}, -\frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2}, -\frac{1}{2} & 0 & \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) + 1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} & 1 & 0 \\ -\frac{1}{2}, \frac{1}{2} & 0 & -1 & \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) + 1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} & 0 \\ -\frac{1}{2}, -\frac{1}{2} & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{16}$$

To obtain braid generators $b\left(j_1=\frac{1}{2},\, j_2=\frac{1}{2}\right)$, choose the appropriate permutation matrix. For $\hat{P}^{j_1=\frac{1}{2},j_2=\frac{1}{2}}$, $\hat{P}^{j_1=\frac{1}{2},j_2=\frac{1}{2}}$ maybe

$$\hat{P}^{j_1=\frac{1}{2},j_2=\frac{1}{2}}=\begin{bmatrix}1&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&1\end{bmatrix}. \tag{17}$$

We apply this permutation matrix in the construction of the braid generator

$$\left(\hat{R}^{j_1,j_2}\right)_{m_1,m_2}^{n_1,n_2}=\left(\hat{P}^{j_1,j_2}\right)_{m_1,m_2}^{m_1',m_2'}\lim_{u\to 0}=\frac{(R^{\frac{1}{2},\frac{1}{2}})_{m_1,m_2}^{n_1,n_2}(u\to 0)}{(R^{\frac{1}{2},\frac{1}{2}})_{\uparrow,\uparrow}^{\uparrow,\uparrow}(u\to 0)}, \tag{18}$$

as a result of which we obtain an explicit form of the $\hat{R}^{j_1=\frac{1}{2},j_2=\frac{1}{2}}$ - matrix for the parametrized eight-vertex model

$$\frac{(R^{\frac{1}{2},\frac{1}{2}})_{m_1,m_2}^{n_1,n_2}(u\to 0)}{(R^{\frac{1}{2},\frac{1}{2}})_{\uparrow,\uparrow}^{\uparrow,\uparrow}(u\to 0)})=\left[\begin{array}{ccccc} m_1,m_2/n_1,n_2 & \frac{1}{2},\frac{1}{2} & \frac{1}{2},-\frac{1}{2} & -\frac{1}{2},\frac{1}{2} & -\frac{1}{2},-\frac{1}{2}\\ \frac{1}{2},\frac{1}{2} & 1 & 0 & 0 & 1\\ \frac{1}{2},-\frac{1}{2} & 0 & 1 & \frac{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})+1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} & 0\\ -\frac{1}{2},\frac{1}{2} & 0 & \frac{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})+1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} & & 0\\ -\frac{1}{2},-\frac{1}{2} & 1 & 0 & 0 & 1\end{array}\right]. \tag{19}$$

A similar construction $\hat{R}^{\frac{1}{2},\frac{1}{2}}$ for $\frac{1}{2},\frac{1}{2}$ is the transposed matrix $\hat{R}^{\frac{1}{2},\frac{1}{2}}$. The explicit form of the identity matrix can be written as

$$\hat{R}^{j_1,j_2}\left[\hat{R}^{j_1,j_2}\right]^{-1}=\hat{R}^{j_1,j_2}\left[[\hat{R}^{j_2,j_1}]^{\mathrm{T}}\right]^{-1}=\Im, \tag{20}$$

$$\hat{R}^{j_1,j_3}\left[\hat{R}^{j_1,j_3}\right]^{-1}=\hat{R}^{j_1,j_3}\left[[\hat{R}^{j_3,j_1}]^{\mathrm{T}}\right]^{-1}=\Im, \tag{21}$$

$$\hat{R}^{j_2,j_3}\left[\hat{R}^{j_2,j_3}\right]^{-1}=\hat{R}^{j_2,j_3}\left[[\hat{R}^{j_3,j_2}]^{\mathrm{T}}\right]^{-1}=\Im. \tag{22}$$

Thus, we have the opportunity to represent matrix forms $b_i[j_1,j_2]$ of groupoids that satisfy the constructions of the braid generators

$$b_i[j_1,j_2][b_i^{j_1,j_2}]\left([b_i^{j_1,j_2}]\right)^{-1}\Im, \tag{23}$$

$$b_i[j_2,j_1][b_i^{j_2,j_1}]\left([b_i^{j_2,j_1}]\right)^{-1}\Im, \tag{24}$$

in this form

$$b_i[j_1,j_2]\ \underbrace{\Im\times\Im\times\ldots\times\Im}_{i-1}\times\left(\hat{R}^{j_1,j_2}\right)^{-1}\times\Im_{i+2}, \tag{25}$$

$$b_i[j_2,j_1]\ \underbrace{\Im\times\Im\times\ldots\times\Im}_{i-1}\times\left(\hat{R}^{j_2,j_1}\right)^{-1}\times\Im_{i+2}. \tag{26}$$

As a result, using the matrix representation, for any word of the braid A, the closure of which will give multicomponent links at the knot. Similarly to the link invariant formula (5), we present the following formula for the fermionic link invariant, in which the components of site $\overset{\approx}{\alpha}_{j_1,j_2}(A)$ have the same spins. The invariant of a multi-colored knot (up to a general deformation coefficient of $q^{\frac{1}{2}}$) for a four-component link, where the components of the knot have the same spins, obtained by closing any r-strand word of braid 10_{136} is given by the formula A

$$\alpha_{j_1,j_2}[A(10_{136})] = q^{\frac{1}{2}C}\tilde{\alpha}_{j_1,j_2}(A) = q^{\frac{1}{2}C}\prod\left(\tau_j\overline{\tau}_j\right)^{-\frac{l_i}{2}} Tr\{HA\}, \tag{27}$$

where the first factor derives the conditioning factor from q with an integer of C, which depends on the spins, bending and the number between the constituent knots of the engagement. l_i's is the number of strands in the braid generator, that is, when spin j_i is in a 5-strand braid A, where $\sum_{i=1}^{n} l_i = r$. In addition, the matrix representation of H depends on the order of such repeating spins occurring in the r-strand braid. This article describes a 5-strand braid obtained from a 10_{136} knot, with a j weave on all strands, which implies

$$H = h_{j_1} \otimes h_{j_2}. \tag{28}$$

Coming back to the matrix operations for the word of braid A, we use formula (11), following the sequence of closure of each link in the braid, where each strand has the same spins mentioned above. From the above definition, h_{j_i}'s (6), τ_{j_i} and $\overline{\tau}_{j_i}$'s (7). In the next section, we will calculate in detail the invariant of knot 10_{136} with the same link.

2. Knot invariant with fermionic junction

At this point of the work, we will find the invariant of a knot with links with spins $j_1 = j_2 = \frac{1}{2}$, for this we first need to write the word for a braid of n strands that tracks the spins. For knot 10_{136}, consisting of the links obtained by closing the five-strand braid and the matrix representation looks like this

$$A(10_{136}) = b_5 b_3^{-1} b_4 b_5^{-1} b_2^{-1} b_5 b_2 b_4 b_1 b_2, \tag{29}$$

and $H = h_{\frac{1}{2}} \otimes h_{\frac{1}{2}}$ will give

$$\alpha_{\frac{1}{2},\frac{1}{2}}[A(10_{136})] = \frac{k\delta\eta^{\frac{5}{2}}}{\psi}, \tag{30}$$

where

$$k = q^2,$$
$$\eta = \left(\left(1+q+q^2+q^3\right)\left(1+q+q^2+q^3\right)\right),$$
$$\psi = \left(\left(-1+q\right)^8\left(1+q\right)^2\left(1+q\right)^2\right),$$
$$\delta = 256 - 2048q + 6928q^2 - 96q^{\frac{5}{2}} - 12176q^3 + 320q^{\frac{7}{2}} + 10352q^4 - 472q^{\frac{9}{2}} + 596q^5 + $$
$$+ 464q^{\frac{11}{2}} - 5429q^6 - 464q^{\frac{13}{2}} - 596q^7 + 472q^{\frac{15}{2}} + 10352q^8 - 320q^{\frac{17}{2}} - 12176q^9 + \quad ,$$
$$+ 96q^{\frac{19}{2}} + 6928q^{10} - 2048q^{11} + 256q^{12}$$

which is consistent with the Jones polynomial calculated on the basis of SU (2) Chern-Simons theory, braid generators, and a fermionic parametrized eight-vertex model.

3. Results and discussion

The paper discusses the construction of a braid group representation using the weighted Boltzmann coefficients of the eight-vertex model, where adjacent edges have different spins. In particular, a matrix representation of braid generators from eight vertex models has been developed. Using the Yang-Baxter formula for the knot invariants, the Jones polynomial for the knot 10_{136} is explicitly evaluated. Finally, the procedure for determining a new representation of braid generators from a R-matrix associated with a parametrized vertex model with elliptic functions is generalized, and a new knot invariant 10_{136} is obtained when studying knots and connections using the Chern-Simons theory. Based on these results, the author comes to the conclusion that the knot invariant is proportional to the two-particle Jones polynomial, where the group representation SU (2) is used, which has state spins on the edges crossing the vertex. In conclusion, the process of finding the knot invariant is described, which is used to find the partition function of the previously known eight-vertex model.

Conclusion

In the course of work on the article, a completely new R-matrix of a two-particle parameterized eight-vertex model with the identical links was obtained from the representations of the braid group for the only knot 10_{136} in the Rolfsen knot table, which has a strand index less than the width of its minimum braid. In addition, the R-matrices defined on braids and constructed by the Markov theorem and the Reidmeister motions are obtained with exact results for the new parametrized eight-vertex model.

Further, we saw that the eight-vertex model is constructed as a modified extension of the integrable six-vertex model and in the three-dimensional positional space elementary particles with half-integer spin are one-dimensional representations of the permutation group acting on the space of wave functions. As a result, it was found that the knot invariant of site 10_{136} is proportional to the two-particle Jones polynomial, where the group representation SU (2) is used, which has state spins j_1, j_2 on the edges intersecting the vertex. In conclusion, the process of finding the knot invariant, which is used to find the partition function of the previously known eight-vertex model, is described.

Further research could fruitfully solve problems in knot theory and statistical mechanics, including the Jones polynomial. Therefore, the problem of graph isomorphism in knot theory and the statistical system may turn out to be an important area for future research.

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