ON CALCULATION METHODS FOR THE MODEL OF PLATES BENDING

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The article is devoted to the study of plate bending problems, which are of great applied importance and are found everywhere in various branches of science and technology. In this article the structure of the calculation methods is described, their main components are highlighted; the classical approach of calculating rectangular plates hinged supported on two parallel sides and with arbitrary boundary conditions on each of the other two sides is characterized. The mathematical apparatus of the method of trigonometric series is presented in the volume necessary for calculating the plates. Special cases of the calculation for the bending of a rectangular plate by the Levi method are given. This article is focused mainly on mechanics, physicists, engineers and technical specialists.

Keywords: bending of a rectangular plate, plate deflection function, boundary conditions of the plate, equation of S. Germain, Navier solution, Levy solution.

Introduction

Now plates are widely used in various fields of science and technology – in mechanics, physics, chemistry, construction, engineering, instrumentation, aviation, shipbuilding, etc. This is due to the fact that the inherent lightness and forms rationality of thin-walled structures are combined with their high bearing capacity, efficiency and good manufacturability.

The plate can be applied as an independent structure or can be part of the used lamellar system. For example, in the construction plates have all kinds of applications in the form of floorings and wall panels, reinforced concrete slabs to cover industrial and residential buildings, slabs for the foundations of massive structures, etc. Therefore, knowledge of the theory for rectangular plates bending and of classical methods for calculating them is necessary for a modern engineer (Fig. 1).
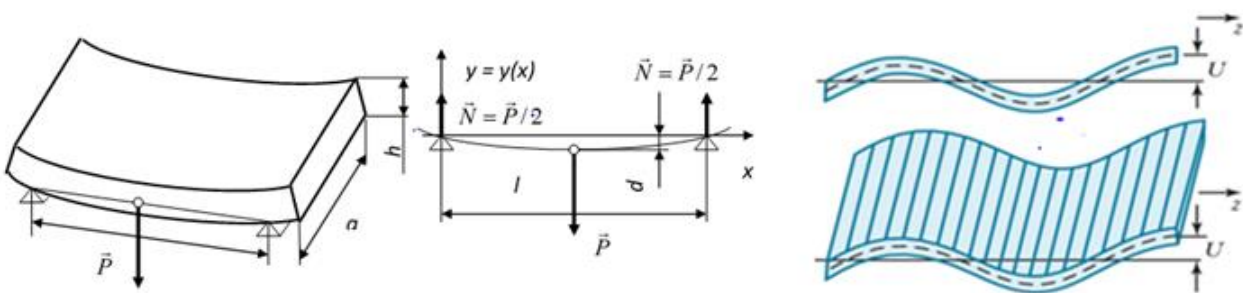

Fig.1. Plate bending. Bending waves

One of the elements of thin-walled spatial systems is a rectangular plate, which has numerous independent applications. An example of a rectangular plate, clamped with one edge, is a vertical panel, and an example of a plate, elastically clamped with three edges, is the wall of a rectangular reservoir. It should be noted that thin plates are a very extensive type of plates and are more often used in many fields of science and technology (Fig. 2).
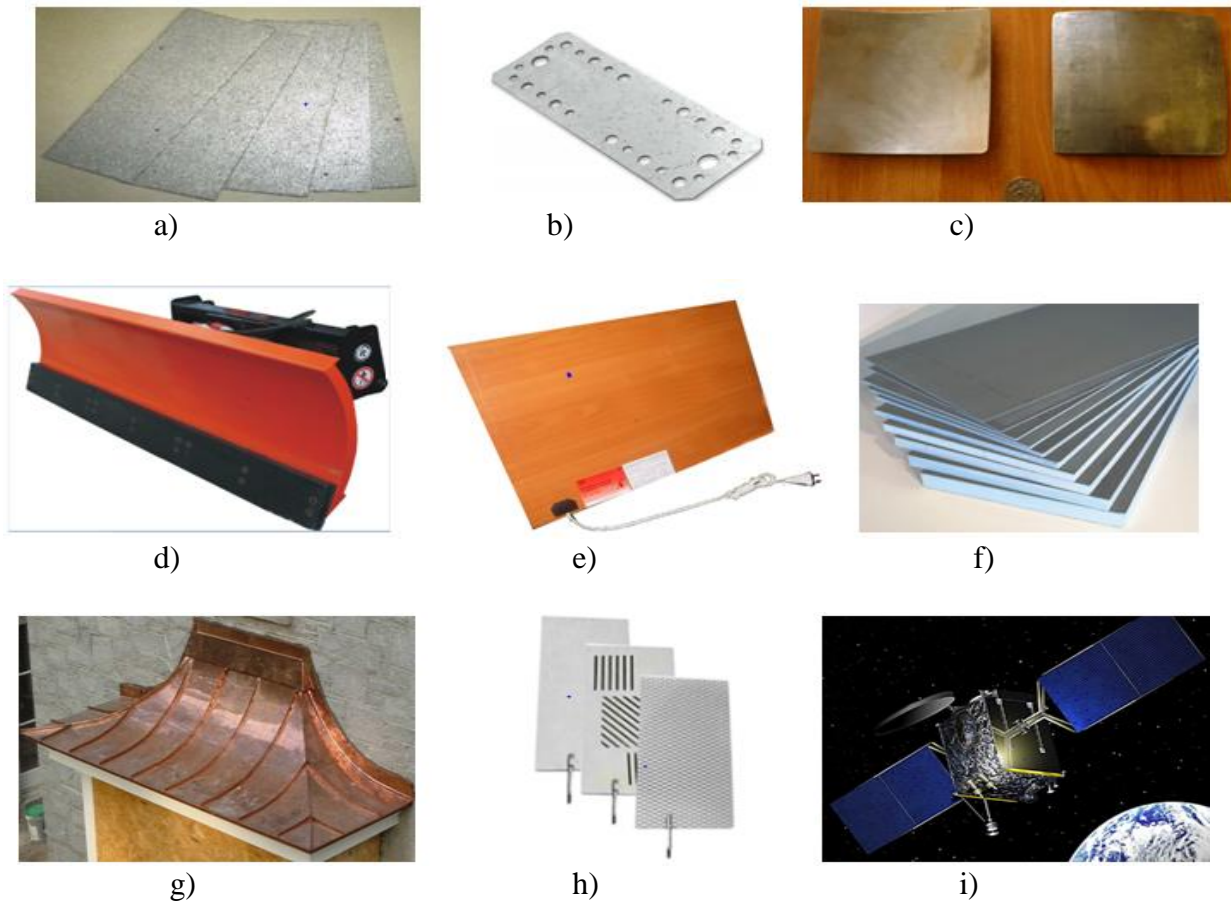
Fig.2. Application of plates:
a) contact plates; b) mounting plate; c) titanium plates of body armor; d) technical plate for road equipment;
e) wall-mounted film heater; f) building plates and slabs; g) copper sheets and plates roof;
h) plates in an alkaline apparatus of water ionization; i) Earth satellite

Many analytical and numerical calculation methods are used to study the problems of plate bending [1- 3]. An exact solution in analytical form for such problems is possible only in some particular cases of the geometrical type of the plate, the load and the conditions for its fixation on the supports, therefore, for engineering practice, approximate, but sufficiently accurate methods for solving the considered boundary value problem are of special importance.

When considering the plate bending problems, the methods of double and single trigonometric series are the most interesting because of connection with their possible numerical implementation in the Maple software package [4].

1. Navier solution

For a rectangular plate ($0 \leq x \leq a$, $0 \leq y \leq b$), hinge supported around the whole contour, we are looking for the desired function $W(x, y)$ of the plate deflections in the form [5]

$$W(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \omega_n x \cdot \sin \sigma_m y , \qquad (1)$$

where A_{nm} are yet unknown coefficients, and $\omega_n = \dfrac{n\pi}{a}$; $\sigma_m = \dfrac{m\pi}{b}$.

The solution in the form of (1) is possible because (1) satisfies the boundary conditions of the hinge support on the plate contour. The given load $f(x, y)$ is also decomposed into a similar trigonometric series

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f_{nm} \sin \omega_n x \cdot \sin \sigma_m y, \tag{2}$$

where coefficients f_{mn} are determined by the formula

$$f_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \omega_n x \cdot \sin \sigma_m y \, dx \, dy.$$

In the particular case of a uniformly distributed load of intensity q we obtain

$$f_{nm} = \frac{16}{nm\pi^2} q.$$

Under the action of the concentrated force P, applied at the point of the plate with the coordinates $x = c, y = d$, we have

$$f_{nm} = \frac{4}{ab} P \sin \omega_n c \cdot \sin \sigma_m d.$$

Substituting the expressions (1) and (2) into the basic resolving equation of S. Germain

$$D \Delta\Delta W = f(x, y), \tag{3}$$

where D is the cylindrical rigidity of the plate, $\Delta\Delta W$ is the biharmonic operator, we find the values A_{mn}. After substituting the values A_{mn} in (1), we obtain that the plate deflections are determined by the formula

$$W(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{f_{nm}}{(\omega_n^2 + \sigma_m^2)^2 D} \sin \omega_n x \cdot \sin \sigma_m y.$$

An example of calculating a square plate $(a = b)$, loaded with a uniformly distributed load q with Poisson's coefficient $v = 0,3$ is given in [6].

2. Levy solution

We consider the case of a plate ($0 \le x \le a$, $0 \le y \le b$), in which only two opposite edges have a hinge support (for example, $x = 0$ and $x = a$) and the other two edges have arbitrary boundary conditions.

We present the desired function of plate deflections $W(x, y)$ in the form [7]

$$W(x, y) = \sum_{n=1}^{\infty} Y_n \sin \omega_n x, \tag{4}$$

where $Y_n = Y_n(y)$ is an unknown function, which is chosen so that expression (4) satisfies the resolving equation S. Germain (3) and boundary conditions on the edges $y = 0$ and $y = b$.

It is obvious that expression (4) satisfies the boundary conditions of hinge support, which are given on the sides $x = 0$, $x = a$ of the plate.

We present the load function $f(x, y)$ in a form of a analogous trigonometric series

$$f(x, y) = \sum_{n=1}^{\infty} f_n(y) \sin \omega_n x , \tag{5}$$

where

$$f_n(y) = \frac{2}{a} \int_o^a f(x, y) \cdot \sin \omega_n x \, dx. \tag{6}$$

Substituting formulas (4) and (5) into the basic differential equation (3), we obtain

$$Y_n^{IV} - 2\omega_n^2 Y_n'' + \omega_n^4 Y_n = \frac{f_n}{D} . \tag{7}$$

The ordinary differential equation (7) allows us to determine an unknown function Y_n for any number n of expansion. Its general solution can be written as

$$Y_n(y) = A_n \cdot ch \omega_n y + B_n \cdot sh \omega_n y + C_n \cdot y \cdot ch \omega_n y + D_n \cdot y \cdot sh \omega_n y + \varphi_n(y) , \tag{8}$$

where A_n, B_n, C_n, D_n is arbitrary integration constants, and φ_n is a partial integral depending on the type f_n and, therefore, on a given external load f.

To determine the four integration constants A_n, B_n, C_n, D_n, the boundary conditions defined at the edges of the plate $y = 0$, $y = b$ are used, and this boundary conditions, of course, can be different. In the general case, this leads to the solving a system of algebraic equations with respect to unknowns A_n, B_n, C_n, D_n.

After finding the coefficients A_n, B_n, C_n, D_n and determining the function $Y_n(y)$ by the formula (8), the plate deflections can be found by the formula (4) in the form of a series, so bending moments, torque, as well as, transverse forces will be written as

$$M_x(x, y) = -D \sum_{n=1}^{\infty} (\nu Y_n'' - \omega_n^2 Y_n) \sin \omega_n x, \quad M_y(x, y) = -D \sum_{n=1}^{\infty} (Y_n'' - \nu \omega_n^2 Y_n) \sin \omega_n x,$$

$$M_{xy}(x, y) = -D(1 - \nu) \sum_{n=1}^{\infty} \omega_n Y_n' \cos \omega_n x, \tag{9}$$

$$Q_x(x, y) = -D \sum_{n=1}^{\infty} \omega_n (Y_n'' - \omega_n^2 Y_n) \cos \omega_n x, \quad Q_y(x, y) = -D \sum_{n=1}^{\infty} (Y_n''' - \omega_n^2 Y_n') \sin \omega_n x.$$

3. The case of uniformly distributed load

Consider the case of a uniformly distributed load of the constant intensity $f = q = const$. Using the formula (5), (6) we obtain

$$q_n = \begin{cases} 0; & n = 2m, \quad m = 1, 2,... \\ \dfrac{4q}{n\pi}; & n = 2m - 1, \quad m = 1, 2,... \end{cases} \tag{10}$$

Then, taking into account (10), the partial integral of equation (7) can be written as

$$\varphi_n = \begin{cases} 0; & n = 2m, \quad m = 1, 2,... \\ \dfrac{4q}{n\pi \omega_n^4 D}; & n = 2m - 1, \quad m = 1, 2,... \end{cases} \tag{11}$$

It can be seen from (11) that for even n, the homogeneous differential equation (7) has only trivial solution, so in the case of a uniformly distributed load of constant intensity the deflection function $W(x, y)$ takes the form

$$W(x, y) = \sum_{m=1}^{\infty} \left[A_{2m-1} ch\omega_{2m-1}y + B_{2m-1} sh\omega_{2m-1}y + y\left(C_{2m-1} ch\omega_{2m-1}y + D_{2m-1} sh\omega_{2m-1}y\right) + \right.$$

$$\left. + \frac{4q}{\pi D(2m-1)\omega_{2m-1}^4} \right] \cdot \sin\omega_{2m-1}x , \tag{12}$$

where the coefficients A_n, B_n, C_n, D_n depend on the given boundary conditions of the plate edges $y = 0$ and $y = b$.

4. Particular cases

As is known, the mathematical model of the plate is completely determined by the deflection function W, and, as shown above, to find the deflection function, it is only necessary to determine the four integration constants A_n, B_n, C_n, D_n, which are found from the boundary conditions. This conditions are given at the edges of the plate $y = 0$ and $y = b$.

Obviously, various approximate methods can be used to find constants A_n, B_n, C_n, D_n. It depends on what degree of accuracy is required in solving a particular practical problem. In addition, it should be borne in mind that the deflection function is defined as an infinite series, finding the sum of which is not always a simple problem. Therefore, it is often necessary to limit ourselves to the finite number of the first members of the series (4) for the deflection function, it also reduces the accuracy of the desired solution.

At the same time, finding analytic expressions for the constants A_n, B_n, C_n, D_n allows us to obtain an analytical expression (formula) for the function of the deflections. And then the function of deflections can be set with the accuracy, which is necessary to solve a particular problem, only limiting the required number of members of the series (4).

We consider finding the integration constants A_n, B_n, C_n, D_n under various boundary conditions on the edges of the plate $y = 0$ and $y = b$ in some particular cases. We show how the coefficients A_n, B_n, C_n, D_n are calculated before analytical expressions are obtained for them.

If we assume that the edges of the plate ($y = 0$ and $y = b$), parallel to the axis x, have a hinge support, then we come to the previously considered Navier solution.

In the case when one of the sides of the plate parallel to the axis x is rigidly pinched and the other side is free, with a uniformly distributed load of constant intensity $f = q$, the integration constants are presented in [7].

Consider this particular case in a more general form, namely, for any kind of external load f. We assume that the side $y = 0$ is free, and the side $y = b$ is rigidly pinched, then the boundary conditions are written as

$$\left(\frac{\partial^2 W}{\partial y^2} + v\frac{\partial^2 W}{\partial x^2} \right)\bigg|_{y=0} = 0, \qquad \left[\frac{\partial^3 W}{\partial y^3} + (2-v)\frac{\partial^3 W}{\partial x^2 \partial y} \right]\bigg|_{y=0} = 0 ; \tag{13}$$

$$W\big|_{y=b} = 0, \qquad \frac{\partial W}{\partial y}\bigg|_{y=b} = 0 . \tag{14}$$

From the boundary conditions (13), (14) and taking into account (4) and (8) we obtain a system

of algebraic equations for determining the coefficients A_n, B_n, C_n, D_n

$$A_n \left[ch\omega_n b - \frac{1-\nu}{2} \omega_n b sh\omega_n b \right] + B_n \left[\frac{1-\nu}{1+\nu} \omega_n b ch\omega_n b + sh\omega_n b \right] = g_1,$$

$$\frac{A_n}{2} \left[-(1-\nu)\omega_n^2 b ch\omega_n b + (1+\nu)\omega_n sh\omega_n b \right] + \frac{B_n}{1+\nu} \left[2\omega_n ch\omega_n b + (1-\nu)\omega_n^2 b sh\omega_n b \right] = g_2,$$

$$-\frac{1-\nu}{1+\nu} \omega_n \cdot B_n + C_n = g_3,$$

$$\frac{1-\nu}{2} \omega_n \cdot A_n + D_n = g_4,$$

where

$$g_1 = -b \cdot (ch\omega_n b \cdot g_3 + sh\omega_n b \cdot g_4) - \varphi_n(b),$$

$$g_2 = -(ch\omega_n b + b\omega_n sh\omega_n b) \cdot g_3 - (sh\omega_n b + b\omega_n ch\omega_n b) \cdot g_4 - \varphi_n'(b).$$

$$g_3 = \frac{1}{(1+\nu)\omega_n^2} \left[(2-\nu)\omega_n^2 \varphi_n'(0) - \varphi_n'''(0) \right],$$

$$g_4 = \frac{1}{2\omega_n} \left[\nu\omega_n^2 \varphi_n(0) - \varphi_n''(0) \right].$$

From the resulting system of equations we find analytical expressions for the coefficients A_n, B_n, C_n, D_n

$$A_n = 2 \frac{\left[2g_3 - (1-\nu)bg_4 \right]ch\omega_n b + \left[(1-\nu)\omega_n bg_3 - \frac{1+\nu}{\omega_n} g_4 \right]sh\omega_n b}{[4 + (1-\nu)^2 \omega_n^2 b^2]ch^2\omega_n b - [(1-\nu)^2 \omega_n^2 b^2 + \nu + 1]sh^2\omega_n b},$$

$$B_n = (1+\nu) \frac{\left[(1-\nu)\omega_n bg_3 + \frac{2}{\omega_n} g_4 \right]ch\omega_n b - [(1+\nu)g_3 + (1-\nu)bg_4]sh\omega_n b}{[4 + (1-\nu)^2 \omega_n^2 b^2]ch^2\omega_n b - [(1-\nu)^2 \omega_n^2 b^2 + \nu + 1]sh^2\omega_n b}, \qquad (15)$$

$$C_n = g_2 + (1-\nu)\omega_n \frac{\left[(1-\nu)\omega_n bg_3 + \frac{2}{\omega_n} g_4 \right]ch\omega_n b - [(1+\nu)g_3 + (1-\nu)bg_4]sh\omega_n b}{[4 + (1-\nu)^2 \omega_n^2 b^2]ch^2\omega_n b - [(1-\nu)^2 \omega_n^2 b^2 + \nu + 1]sh^2\omega_n b},$$

$$D_n = g_1 - (1-\nu)\omega_n \frac{\left[2g_3 - (1-\nu)bg_4 \right]ch\omega_n b + \left[(1-\nu)\omega_n bg_3 - \frac{1+\nu}{\omega_n} g_4 \right]sh\omega_n b}{[4 + (1-\nu)^2 \omega_n^2 b^2]ch^2\omega_n b - [(1-\nu)^2 \omega_n^2 b^2 + \nu + 1]sh^2\omega_n b},$$

Now we consider the case when one of the sides of the plate (for example, a side $y = 0$) parallel to the x axis is supported by an elastic contour, and the other side is rigidly pinched. The elastic contour may be, for example, a beam, bending under the action of pressures applied to it. The boundary conditions on the side $y = 0$ have the form

$$\left(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \right)\bigg|_{y=0} = 0, \quad D\left[\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial x^2 \partial y} \right]\bigg|_{y=0} = \left(EJ \frac{\partial^4 W}{\partial x^4} \right)\bigg|_{y=0},$$

where EJ is the rigidity of the beam.

Analytical expressions for the coefficients A_n, B_n, C_n, D_n are obtained in the same way and have the form

$$A_n = \tilde{g}_1 -$$

$$-\frac{2}{(1-v)\omega_n}\cdot\frac{D(1-v)\left\{-\left[2ch\omega_n b+b\omega_n(1-v)sh\omega_n b\right]\tilde{g}_3+\left[\dfrac{1+v}{\omega_n}sh\omega_n b+b(1-v)ch\omega_n b\right]\tilde{g}_4\right\}}{\left(b\tau\omega_n+\dfrac{4D}{\omega_n}\right)ch^2\omega_n b+\left[Db(1-v)^2-\tau\right]sh\omega_n b\cdot ch\omega_n b-\left[\dfrac{D}{\omega_n}(1+v)^2+b\tau\omega_n\right]sh^2\omega_n b},$$

$$B_n=\tilde{g}_2+$$

$$+\frac{2EJ}{D(1-v)^2}\cdot\frac{D(1-v)\left\{-\left[2ch\omega_n b+b\omega_n(1-v)sh\omega_n b\right]\tilde{g}_3+\left[\dfrac{1+v}{\omega_n}sh\omega_n b+b(1-v)ch\omega_n b\right]\tilde{g}_4\right\}}{\left(b\tau\omega_n+\dfrac{4D}{\omega_n}\right)ch^2\omega_n b+\left[Db(1-v)^2-\tau\right]sh\omega_n b\cdot ch\omega_n b-\left[\dfrac{D}{\omega_n}(1+v)^2+b\tau\omega_n\right]sh^2\omega_n b}$$

(16),

$$C_n=\frac{\left[\tau\omega_n ch\omega_n b-D(1-v^2)sh\omega_n b\right]\tilde{g}_3+\left[\dfrac{2D}{\omega_n}(1-v)ch\omega_n b-\tau\cdot sh\omega_n b\right]\tilde{g}_4}{\left(b\tau\omega_n+\dfrac{4D}{\omega_n}\right)ch^2\omega_n b+\left[Db(1-v)^2-\tau\right]sh\omega_n b\cdot ch\omega_n b-\left[\dfrac{D}{\omega_n}(1+v)^2+b\tau\omega_n\right]sh^2\omega_n b},$$

$$D_n=\frac{D(1-v)\left\{-\left[2ch\omega_n b+b\omega_n(1-v)sh\omega_n b\right]\tilde{g}_3+\left[\dfrac{1+v}{\omega_n}sh\omega_n b+b(1-v)ch\omega_n b\right]\tilde{g}_4\right\}}{\left(b\tau\omega_n+\dfrac{4D}{\omega_n}\right)ch^2\omega_n b+\left[Db(1-v)^2-\tau\right]sh\omega_n b\cdot ch\omega_n b-\left[\dfrac{D}{\omega_n}(1+v)^2+b\tau\omega_n\right]sh^2\omega_n b},$$

where

$$\tilde{g}_1=\frac{1}{(1-v)\omega_n^2}\left[v\omega_n^2\varphi_n(0)-\varphi_n''(0)\right],$$

$$\tilde{g}_2=-\frac{EJ}{D(1-v)^2}\left[v\omega_n\varphi_n(0)-\frac{1}{\omega_n}\varphi_n''(0)\right]+\frac{1}{(1-v)\omega_n}\left[\frac{1}{\omega_n^2}\varphi'''(0)-(2-v)\varphi'(0)-\frac{EJ}{D}\omega_n^2\varphi_n(0)\right],$$

$$\tilde{g}_3=\left[\frac{EJ}{D(1-v)^2}sh\omega_n b-\frac{1}{(1-v)\omega_n}ch\omega_n b\right]\cdot\left[v\omega_n\varphi_n(0)-\frac{1}{\omega_n}\varphi''(0)\right]-$$

$$-\frac{sh\omega_n b}{(1-v)\omega_n}\left[\frac{1}{\omega_n^2}\varphi_n'''(0)-(2-v)\varphi_n'(0)-\frac{EJ}{D}\omega_n^2\varphi_n(0)\right]-\varphi_n(b),$$

$$\tilde{g}_4=\left[\frac{EJ\omega_n}{D(1-v)^2}ch\omega_n b-\frac{1}{1-v}sh\omega_n b\right]\cdot\left[v\omega_n\varphi_n(0)-\frac{1}{\omega_n}\varphi_n''(0)\right]-$$

$$-\frac{ch\omega_n b}{1-v}\left[\frac{1}{\omega_n^2}\varphi_n'''(0)-(2-v)\varphi_n'(0)-\frac{EJ}{D}\omega_n^2\varphi_n(0)\right]-\varphi_n'(b),$$

$$\tau=2EJ+bD(1-v)^2.$$

Due to the bulkiness of formulas for the determination of the coefficients A_n, B_n, C_n, D_n in the general case, and, consequently, due to the inconvenience and complexity of further use of these formulas, it is recommended that all calculations of the constants A_n, B_n, C_n, D_n be carried out for particular numerical values of a problem in each particular case with given numerical parameters.

Substitution of the found coefficients A_n, B_n, C_n, D_n in (8), (4) and (9) gives the function of plate deflections $W(x, y)$, bending moments and torques, as well as transverse forces in the form of trigonometric series in the each particular case considered above.

In the case of a uniformly distributed load of constant intensity q, the deflection function $W(x, y)$ has the form (12) with coefficients (15) or (16).

Conclusion

Without any difficulty, Levy solution can also be applied to the study for the bending of a plate whose the sides parallel to the axis x have another boundary conditions. Levy solution also extends easily to those cases where the sides of the plate contour, parallel to the axis x, are not quite rigid, but are relatively flexible beams, that bend under the action of the pressures acting on them.

In principle, Levy solution is more accurate than Navier solution, since in it the desired function $W(x, y)$ is approximated by trigonometric functions only in one direction, and in the other direction it is sought precisely from the differential equation (7).

It should be noted that when calculating the plates by analytical methods in the most general formulation: with arbitrary boundary conditions (including elastic), different types of load, complex shapes of plates, with cuts, projections, etc., we have to face with great mathematical difficulties, and in most cases to obtain an analytical solution is not possible. Such a problem can be solved by applying a very efficient finite element method, which is a numerical approximate method for plates, but which gives a sufficiently high accuracy of solutions.

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