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THE AVERAGED LAGRANGIAN OF SOLITONS OF BOSE-EINSTEIN CONDENSATION

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The effect of the fluctuation interaction of atoms forming a condensate on solitons of the Bose-Einstein condensation is considered. A short derivation of the Gross-Pitaevsky equation is given. The averaged Lagrangian of solitons of the Bose-Einstein condensation is determined. The obtained results give an opportunity to investigate the solitons dynamics under the action of various perturbations.

Keywords: Bose-Einstein condensate, Gross-Pitaevskii equation, averaged Lagrangian.

Introduction

Increased interest in studies of Bose - Einstein condensation is due to the fact that this phenomenon plays an important role in physical systems consisting of strongly interacting nanoparticles [1-2]. Probably, it also causes high-temperature superconductivity.

In this paper, a great deal of attention is paid to studying the influence of the fluctuation interaction of atoms forming the condensate. We present the short derivation of the Gross - Pitaevskii equation and find the averaged Lagrangian of solitons in the Bose - Einstein condensate.

1. Short derivation of the Gross - Pitaevskii equation

As is known, the quantum state of Bose - Einstein condensate atoms is described by one common wave function because all of them have the minimum possible energy. Therefore, in accordance with quantum mechanics, this wave function is described by the Schrodinger equation:

$$i\eta\frac{\partial\psi}{\partial t} = (\hat{T} + u(r,t) + V_{\mathrm{int}})\psi \tag{1}$$

Here: $\psi(r,t)$ is the wave function of the system of particles under consideration, $\hat{T}$ is the kinetic-energy operator, $u(r,t)$ is the external-field potential, and V_{int} is the interaction potential of atoms forming the condensate.

If one restricts oneself to the case where the external-field potential is isotropic and spherically symmetric (which is used in experiments for studying Bose - Einstein condensation to create the trap potential), it can be written in the form of the following spherical harmonic (or so-called parabolic) potential:

$$u(r) = \frac{k}{2}r^2 \tag{2}$$

where k is the force constant of the harmonic potential (or the effective elasticity coefficient).

In this case, it is convenient to use the spherical coordinate system, in which the operator of kinetic energy has the form:

$$\hat{T} = -\frac{\eta^2}{2m}\cdot\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) \tag{3}$$

Consequently, if the system of atoms is a collection of noninteracting particles, then, in accordance with Eq. (1), we obtain the usual linear Schrodinger equation describing the quantum state of so-called spherical harmonic oscillators.

Therefore, as was shown by Gross and Pitaevskii for the first time [2], to form the Bose - Einstein condensate, atoms must interact, and the interaction potential is defined by the following expression:

$$V_{\text{int}} = \frac{4\pi\eta^2 a_0}{m}|\psi|^2 \tag{4}$$

Here, a_0 is the force constant of the interaction of atoms (or a so-called scattering constant); it is negative for atoms attracting one another and positive for atoms repelling one another.

Thus, on the basis of the above preconditions, the equation describing the wave function of Bose—Einstein condensate atoms can be written in the following form:

$$i\eta\frac{\partial\psi}{\partial t} = -\frac{\eta^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}\psi) + \frac{1}{2}kr^2\psi + \frac{4\pi\eta^2 a_0}{m}|\psi|^2\psi \tag{5}$$

Then we reduce Eq. (5) to the so-called canonical form. To do this, we introduce the following notation:

$$r = A_0\rho, \qquad t = B_0\tau, \qquad \psi = c_0 f. \tag{6}$$

Here, A_0, B_0, and C_0 are the characteristic parameters of the system under consideration, which will be defined below. Then ρ, τ, and *f* denote the dimension- less coordinate, the time, and the wave function.

Taking relations (6) into account and substituting them into (5), we obtain

$$i\frac{\partial f}{\partial\tau} = -\frac{\eta}{2m}\cdot\frac{B_0}{A_0^{\ 2}}\cdot\frac{1}{\rho^2}\cdot\frac{\partial}{\partial\rho}(\rho^2\frac{\partial f}{\partial\rho}) + \frac{k}{2\eta}A_0^{\ 2}B_0\rho^2 f + \frac{4\pi\eta a_0}{m}B_0 c_0^{\ 2}|f|^2 f \tag{7}$$

Then, rewriting the last equation in the form of the canonical Gross—Pitaevskii equation

$$i\frac{\partial f}{\partial\tau} + \frac{1}{\rho^2}\cdot\frac{\partial}{\partial\rho}(\rho^2\frac{\partial f}{\partial\rho}) - \frac{1}{2}\rho^2 f - S|f|^2 f = 0 \tag{8}$$

we obtain the following conditions for determination of the characteristic parameters:

$$\frac{\eta}{2m}\cdot\frac{B_0}{A_0^{\ 2}} = 1, \qquad \frac{k}{\eta}\cdot A_0^{\ 2}B_0 = 1, \qquad S = \frac{4\pi\eta a_0}{m}B_0 c_0^{\ 2}. \tag{9}$$

Hence it follows that the characteristic distance A_0, and the time B_o are determined by the masses *m* and the force constant of *k* atoms forming the Bose condensate:

$$B_0 = \sqrt{\frac{2m}{k}}, \; A_0 = \sqrt{\frac{\eta}{kB_0}}. \tag{10}$$

And finally, in accordance with the last condition (9), we find that

$$c_0^{\ 2} = \frac{mS}{4\pi\eta a_0 B_0^{\ 2}}, \tag{11}$$

This formula expresses the density of Bose-condensate particles. Consequently, $c_0^{\ 2}A_0^{\ 3}$ characterizes the number of atoms passed to the condensed state.

2. Averaged Lagrangian of solitons in the Bose - Einstein condensate

As is known, to obtain the equation of motion (irrespective of the fact whether this approximation is classical or quantum) of any physical system, it is necessary to determine its Lagrange function, or Lagrang ian. Then, in accordance with the principle of least action (or the variational approach), the equations of motion for the system are described by the Euler - Lagrange equations. Therefore, the aim is to find the Lagrangian for solitons in the Bose - Einstein condensate.

Then we use the method of the averaged Lagrangian, which was proposed and developed in [4 - 6], taking into account that the Gross - Pitaevskii equation (8) includes the following “conservation laws”, or the integrals of motion:

$$I_1 = 4\pi\int_0^\infty \rho^2 |f|^2 d\rho \tag{12}$$

$$I_2 = 4\pi\int_0^\infty \rho^2 d\rho\left[\left|\frac{\partial f}{\partial\rho}\right|^2 + \frac{1}{4}\rho^2|f|^2 + \frac{1}{2}S|f|^4\right]. \tag{13}$$

Here, the first integral expresses the number of soli- tons in the Bose - Einstein condensate and the second, their energy. It follows from (12) and (13) that, if the solitons are not affected by an external force, then their number and energy are conserved.

When external fields (or perturbation, as they say) act on the collection of atoms forming the Bose condensate, the parameters of the solitons are changed. In particular, the wave function of solitons, which has the following form, can be altered:

$$f(\rho,\tau) = A(\tau)\exp\left[-\frac{\rho^2}{2a^2(\tau)} + \frac{i}{2}b(\tau)\rho^2 + i\varphi(\tau)\right]. \tag{14}$$

Here, $A(\tau), a(\tau), b(\tau)$ and $\varphi(\tau)$ expresses the respective amplitude, width, local frequency, and the phase of the wave function of solitons (or, simply speaking, the parameters of the soliton solution (14)).

It is natural that, under the action of rather strong perturbations, which are functions of coordinates and time, determination of the solution of the Gross - Pitaevskii equation becomes a problem. But from the physical point of view, the cases where external perturbation is weak are most important; i.e., the analysis of changes in steady-state and soliton solutions is important. It is this case where the method of the averaged Lagrangian is used; it assumes the adiabaticity of perturbations. This means that wave function (14) is conserved, but its parameters are slowly varying time functions.

Before finding the averaged Lagrangian of solitons in the Bose - Einstein condensate, we determine their number and energy using wave function (14) and calculating integrals (12) and (13). We obtain the following expressions for them:

$$I_1 = \pi\sqrt{\pi}A^2a^3 \tag{15}$$

$$I_2 = \frac{\sqrt{\pi}}{4}A^2a\left\{\frac{3}{2}\left[a^4(b^2+\frac{1}{4})+1\right]+\pi SA^4a\right\} \tag{16}$$

Now we pass to determination of the averaged Lagrangian of solitons in the Bose - Einstein condensate:

$$\overline{L} = \int dVL \equiv 4\pi\int_0^\infty \rho^2 d\rho L \tag{17}$$

The Lagrange function L, from which the canonical Gross - Pitaevskii equation (8) is obtained, has the following form (it is not difficult to be convinced of it):

$$L = |f|^2 + \left|\frac{\partial f}{\partial \rho}\right|^2 + \frac{1}{4}\rho^2 |f|^2 + \frac{1}{2}S|f|^4 + \frac{i}{2}\left(f^* \frac{\partial f}{\partial \tau} - f\frac{\partial f^*}{\partial \tau}\right). \tag{18}$$

Taking the last term in expression (18) into account and integrating (17), we obtain the following general expression for the averaged Lagrangian of solitons in the Bose - Einstein condensate:

$$\overline{L} = \frac{\sqrt{\pi}}{4}A^2\left\{\frac{3}{2}\left[a^5b^2 + a + \frac{a^5}{4}\right] + \pi S A^2 a^2\right\} + 2\pi A^2\left[\frac{db}{d\tau}\cdot\frac{3\sqrt{\pi}}{8}a^5 + 2\frac{d\varphi}{d\tau}\cdot\frac{\sqrt{\pi}}{4}a^3\right]. \tag{19}$$

Conclusion

Based on the above studies, we obtained the following results:

1. A brief derivation of the Gross-Pitaevskii equation describing the wave function of the atoms of the Bose-Einstein condensate is given.
2. The averaged Lagrangian of solitons of the Bose-Einstein condensation is found, on the basis of which it is possible to study the dynamics of solitons under the action of various perturbations.

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